I: Thinking about the wave function

In quantum mechanics, the term wave function usually refers to a solution to the Schrödinger equation,

\[ i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x)\Psi(x, t), \]

where \( V(x) \) is the potential energy experienced by a particle of mass \( m \) and \( \Psi(x, t) \) is the wave function in this one-dimensional example.

A. Let’s say you have a system where the wave function is of the form:

\[ \Psi_1(x, t) = f(x)e^{i\omega t} \]

where \( f(x) \) is some real-valued function of \( x \).

1. Is \( |\Psi_1(x, t)|^2 \) real? Is it positive? Do your answers make sense given the physical meaning (as discussed in class) of \( |\Psi_1(x, t)|^2 \)?

2. Does \( \Psi_1(x, t) \) depend on time? Does \( |\Psi_1(x, t)|^2 \) depend on time?

3. Write down an expression for \( \langle x \rangle \). Does it depend on time? Is it real?

Describe in words how you interpret this quantity. Precisely what information do you get from \( \langle x \rangle \)?

4. Write down an expression for \( \langle g(x) \rangle \) where \( g(x) \) is any real-valued function of \( x \). Does it depend on time? Again, how would you physically interpret \( \langle g(x) \rangle \) (hint: think about what you would actually measure)?
B. Now let’s say your system is a bit more complex (pun intended):

$$\Psi_2(x, t) = f(x)e^{i\omega t} + g(x)e^{2i\omega t}$$

where $f(x)$ and $g(x)$ are real functions of $x$ which are orthogonal to each other.

1. Is $|\Psi_2(x, t)|^2$ real? Is it positive? Do your answers make sense given the physical meaning of $|\Psi_2(x, t)|^2$?

2. Does $|\Psi_2(x, t)|^2$ depend on time?

3. Write down an expression for $\langle x \rangle$. Does it depend on time? Describe the difference(s) between this result and the result for section A.3 above.

Even though $f$ and $g$ are unknown functions of $x$, do your best to give a physical description or interpretation of this new result for $\langle x \rangle$ for the state $\Psi_2$.

✓ Check your results with a tutorial instructor.
C. Now, we will deal with a new wave function at a single moment in time, 
\[ \psi_3(x) = \Psi_3(x, t = t_0), \]
represented by the graph below (a sine curve from \( \pi/4 \) to \( 5\pi/4 \) and zero everywhere else).

1. Find a value of A which will normalize \( \psi_3(x) \).

2. Using physical arguments (i.e., without doing the integral), what do you think \( \langle x \rangle \) is? (If you feel uncertain, you can check by doing the integral)

3. We want to find the standard deviation for \( x \) for this system. First, do you think that \( \langle x^2 \rangle \) is larger/the same/smaller (circle one) than \( \langle x \rangle^2 \)? Now, actually calculate \( \langle x^2 \rangle \).

4. What is \( \sigma_x^2 \)? What is the probability that you will find the particle represented by \( \psi_3(x) \) in the range \( \langle x \rangle \pm \sigma_x \)? (Recall that \( \sigma_x^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 \).)
D. Now we somehow create a system where for an instant, the wave function, 
\( \psi_4(x) = \Psi_4(x, t = t_0) \), looks like the graph below.

\[ \psi_4(x) \]

1. Find the value of A which will normalize \( \psi_4(x) \).

2. Using physical arguments (i.e., without doing the integral), what do you think \( \langle x \rangle \) is? (If you feel uncertain, you can check by doing the integral)

3. Estimate \( \langle x^2 \rangle \) and \( \sigma_x \). Indicate on the graph above the range which you think represents \( \langle x \rangle \pm \sigma_x \).
   
   Bonus (i.e., come back to this if you have time after finishing the rest of the tutorial), calculate \( \langle x^2 \rangle \) and \( \sigma_x^2 \).

4. How do you physically interpret \( \sigma_x \)?

5. What are the possible values of a measurement of \( x \) on any of these identical systems? Do you “expect” to measure \( x \) equal to the expectation value of \( x \)?

✓ Check your results with a tutorial instructor.
II: Classical current

A. Consider a thin, insulated wire with a current which depends on the position along the wire. Let the current be given as \( I(x) \), where a positive value of \( I \) represents current flowing to the right.

\[
\begin{array}{c}
  \text{I} \\
  \longrightarrow \\
  \text{a} \quad \text{b} \\
  \longrightarrow +x
\end{array}
\]

Student A defines \( Q_{ab}(t) \) to be the total electric charge in the wire between points a and b (see figure above). Student B points out that since charge cannot be created or destroyed (i.e., charge is conserved), \( Q_{ab} \) cannot be a function of time. You are called in to settle the dispute. Could \( Q_{ab} \) depend on time? What is your reasoning?

B. No matter what you said above, suppose we told you we had set up a situation where at an instant of time, \( t_0 \), we had measured \( I(b) > I(a) \).

1. What does this situation imply about the time dependence of \( Q_{ab} \)?

2. Construct a formula for the time derivative of \( Q_{ab} \) in terms of \( I(a) \) and \( I(b) \)
Useful Formulas

\[ \int_{a}^{b} \sin^2(x - x_0) \, dx = \left( \frac{x - x_0}{2} - \frac{\sin(2(x - x_0))}{4} \right) \bigg|_{a}^{b} \]

\[ \int_{a}^{b} x \sin^2(x - x_0) \, dx = \left( \frac{x^2 - x_0^2}{4} - \frac{\cos(2(x - x_0))}{8} - \frac{x \sin(2(x - x_0))}{4} \right) \bigg|_{a}^{b} \]

\[ \int_{a}^{b} x^2 \sin^2(x - x_0) \, dx = \frac{1}{24} \left( 4x^3 - 6x \cos(2(x - x_0)) + (3 - 6x^2) \sin(2(x - x_0)) \right) \bigg|_{a}^{b} \]