Work and Energy

Energy is difficult to define because it comes in many different forms. It is hard to find a single definition which covers all the forms. But we will define energy in the next chapter, as precisely as possible. For now, let's just list some forms of energy.

Some types of energy:

kinetic energy (KE) = energy of motion
thermal energy = energy of "atomic jiggling"
potential energy(PE) = stored energy of position/configuration
various kinds of PE:

- gravitational
- electrostatic
- elastic (actually a form of electrostatic PE)
- chemical (another form of electrostatic PE)
- nuclear

radiant energy = energy of light
mass energy (Einstein says mass is a form of energy.)

Almost all forms of energy on earth can be traced back to the Sun :

Example: Lift a book (gravitational PE) ← chemical PE in muscles ← chemical PE in food ← cows ← grass ← sun (through photosynthesis) !

Some textbooks say that energy is the ability to do work (not a bad definition, but rather vague). A key idea that we will use over and over again is this: **Whenever work is being done, energy is being changed from one form to another or is being transferred from one body to another. The amount of work done on a system is the change in energy of the system.**

We'll use the symbol W for work and the symbol U for energy. (We will define work later.) The English sentence "**The work done equals the amount of energy transformed"** we can write as

$$W = \Delta U$$

This is called "The First Law of Thermodynamics".

[An aside. Actually the First Law of Thermodynamics is this:
"heat added plus work done equals change in energy" or $Q + W = \Delta U$. (Q is the symbol for heat). In this chapter we won't consider adding heat to a system (like holding a flame under it), so $Q = 0$ and we have just $W = \Delta U$.]

As we'll see later, energy is an extremely useful concept because energy is **conserved.** When we say energy is conserved, we mean that energy cannot be created or destroyed; it can only be transformed from one form to another, or transferred from one body to another. The total amount of energy everywhere is fixed; all we can do is shuffle it around.

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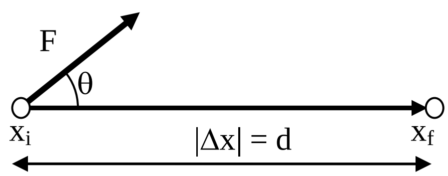
Notice that this is not what people normally mean when they say "Conserve energy." When the power company says "Conserve energy", they really mean "Don't convert the energy stored as chemical potential energy into other forms of energy too quickly." To a scientist, the phrase "conserve energy" is meaningless, because energy is always conserved. You can't NOT conserve energy.

To understand energy and conservation of energy, we must first define some terms: work, kinetic energy (KE), and potential energy (PE). We'll get to PE in the next Chapter. Let's look at work and KE.

Definition of work done by a force: consider an object moving while a constant force $\vec{F}$ is applied to the object. While the force is applied, the object moves along some axis (x-axis, say) through a displacement of magnitude $|\Delta x| = d$.



Notice that the direction of displacement is not the same as the direction of the force, in general.

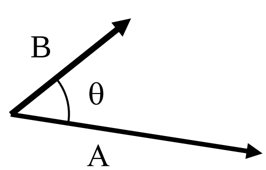
Work done by a force F = $\boxed{W_F \equiv F_x \cdot d = F\cos\theta\ d = F_{||}\ d}$

$F_{||}$ = component of force along the direction of displacement, $W_F = F_{||} \times$ distance

Unit of work: [W] = [F][d] = 1 N·m = 1 joule = 1 J

If the displacement vector is $\Delta\vec{r}$, the work done can be written in terms of the **dot product** as $W_F \equiv \vec{F} \cdot \Delta\vec{r}$

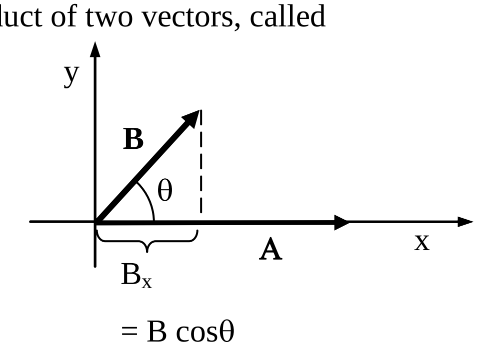
Vector Math interlude:
The dot product of two vectors A and B, "A dot B", is defined as



$$\boxed{\vec{A} \cdot \vec{B} \equiv A B \cos\theta}$$

The dot product of two vectors is a number, not a vector. (Later on, we will see another way to define the product of two vectors, called the "cross-product". The *cross-product* of two vectors is a vector.)

The dot product is the magnitude of one vector (say **A**) times the component of the other vector (**B**) along the direction of the first (**A**). For instance, suppose that we align the x-axis with vector **A**. Then $\vec{A} \cdot \vec{B} = A B \cos\theta = A B_x$.



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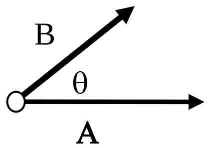
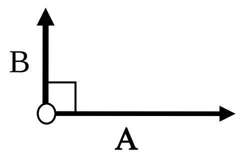
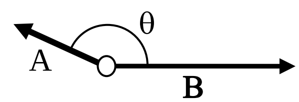
The dot product is positive, negative, or zero depending on the relative directions of the vectors **A** and **B**. When **A** and **B** are at right angles ($\theta = 90^o$), the dot product is zero. When the angle θ is greater then 90^o, then the dot product is negative.



$\theta < 90^o$, **A·B** positive



$\theta = 90^o$, **A·B** = 0



$\theta > 90^o$, **A·B** negative

It is not difficult to prove, from the definition, that ..

- the dot product is commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- the dot product is associative: $\left(\vec{A}+\vec{B}\right)\cdot\vec{C} = \vec{A}\cdot\vec{C} + \vec{B}\cdot\vec{C}$

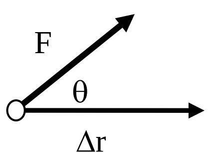
The dot product can be written in terms of the components of the vectors like so:

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

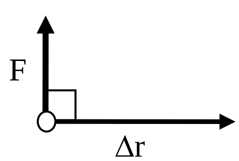
Proof (in 2D) : $\vec{A}\cdot\vec{B} = \left(A_x\hat{i} + A_y\hat{j}\right)\cdot\left(B_x\hat{i} + B_y\hat{j}\right) = A_x B_x\,\hat{i}\cdot\hat{i} + A_x B_y\,\hat{i}\cdot\hat{j} + ... = A_x B_x + A_y B_y$

In the last step, we used the fact that $\hat{i}\cdot\hat{i} = 1$ and $\hat{i}\cdot\hat{j} = 0$.

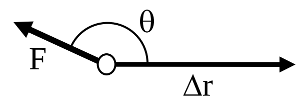
So, the work done by a force F is $W_F \equiv \vec{F}\cdot\Delta\vec{r}$. Work is not a vector, but it does have a sign, (+) or (–). Work is positive, negative, or zero, depending on the angle between the force and the displacement. The formula $W_F = F\,d\cos\theta$ gives the correct sign, because $\cos\theta$ is negative when $\theta > 90$.



$\theta < 90$, W positive



$\theta = 90$, W = 0



$\theta > 90$, W negative

Why do we define work this way? Answer: Whenever work is done, energy is being transformed from one form to another. The amount of work done is the amount of energy transformed. (This is the First Law of Thermodynamics.)

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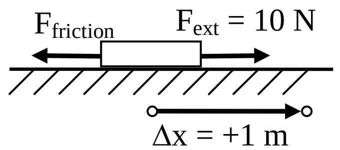
Example of work: Move book at constant velocity along a rough table with a constant horizontal force of magnitude F_{ext} = 10 N (10 newtons). Total displacement is Δx = 1 m.



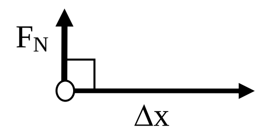
work done *by external force* =

$$W_{Fext} = +F_{ext} \cdot \Delta x = 10\text{ N} \cdot 1\text{ m} = 10\text{ N}\cdot\text{m} = +10\text{ J}$$

Since velocity = constant, $F_{net} = 0$, so $|F_{ext}| = |F_{fric}|$ = 10 N

Work done *by force of friction* = $W_{Ffric} = -|F_{fric}| \cdot |\Delta x| = -10$ J (since $\cos 180^o = -1$)

Work done *by normal force* F_N is zero: $W_{FN} = 0$
(since normal force is perpendicular to displacement, $\cos 90^o = 0$.)



Work done *by the net force* is zero. Since v = constant $\Rightarrow F_{net} = 0 \Rightarrow W_{net} = 0$.

Moral of this example: Whenever you talk about the work done, you must be <u>very careful</u> to specify *which force* does the work.

Definition of kinetic energy (KE) of an object of mass m, moving with speed v:

$$\boxed{KE \equiv \tfrac{1}{2} m v^2}$$

KE > 0 always. An object has a big KE if it is massive and/or is moving fast. KE is energy of motion.

$$\text{Units of KE} = [KE] = kg \cdot \left(\frac{m}{s}\right)^2 = \underbrace{kg \cdot \frac{m}{s^2}}_{\text{units of force }=[m][a]} \cdot m = N \cdot m = J \text{ (joules)}$$

Units of KE = units of work = joules

Example of KE:
Bowling ball (weight mg = 17 lbs, mass m = 7.7 kg) with speed v = 7 m/s (typical bowling speed). $KE = 0.5\,(7.7\text{ kg})\,(7\text{ m/s})^2 \cong 190$ J

Why do we define work and KE like we have? Because work and KE are related by the …

Work-Kinetic Energy Theorem:
The work done by the *net force on a single point-like object* is equal to the change in kinetic energy of that object.

$$\boxed{W_{net} = W_{Fnet} = \Delta KE = KE_f - KE_i}$$

Notice that this is the work done by the total force, the **net** force. The Work-KE Theorem applies in the special cast that the object is “point-like”, meaning the object can be treated like a

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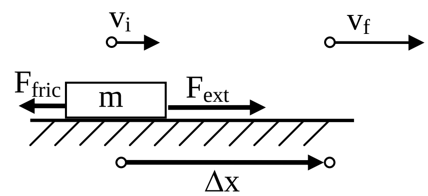
single particle with no deformation and no rotation. (If the object has any moving internal parts, then there is no single speed for the object and the KE of the object is not simply $1/2mv^2$.)

"Proof" of $W_{net} = \Delta KE$. Here we show that the Work-KE Thm is true for a special case. I push a book of mass m along a table with a constant external force of magnitude $F_{ext.}$ The force of friction on the book has magnitude F_{fric}. The book starts with an initial velocity v_i and ends with a final velocity v_f. While the force is applied, the book moves a displacement Δx. We show that $W_{net} = \Delta KE$ in this case.



$F_{net} = F_{ext} - F_{fric}$ (the normal force and force of gravity cancel). $W_{net} = +F_{net}\,\Delta x$

What is ΔKE? KE involve v^2, so we look for a formula involving v^2. Since F_{net} = constant, the acceleration is constant, and so we can use the 1D constant acceleration formula $v^2 = v_0^2 + 2a(x - x_0)$. So we have

$$v_f^2 - v_i^2 = 2a(x_f - x_i) = 2(F_{net}/m)(\Delta x) \quad [\text{ using } a = F_{net}/m]$$

$\Delta KE = KE_f - KE_i = (1/2)mv_f^2 - (1/2)mv_i^2 = (1/2)m \cdot 2(F_{net}/m)(\Delta x) = +F_{net}\,\Delta x = W_{net}$. Done!

Energy was transferred from the surroundings into the KE of the book. We have shown that the Work-KE Theorem is true in this one case, but it turns out to be always true whenever the object can be treated as a single particle. A more complete derivation is given in the Appendix to this chapter.

Notice the Work-KE theorem holds even when friction is involved.

Let's check that the Work-KE Theorem works in a few other special cases:

Example: A book of mass m is dropped from rest and it falls a distance h. The net force is $F_{grav} = mg$. The work done by the net force is $W_{net} = +mg\,|\Delta y| = mgh$. To compute the change in KE, we need the final velocity: $\Delta KE = KE_f - KE_i = KE_f - 0 = (1/2)mv_f^2$. The final velocity we get from our constant acceleration formula: $v_f^2 = v_i^2 + 2a(y_f - y_i) = 0 - 2g(\Delta y) = +2gh$ (notice Δy is negative). So $(1/2)mv_f^2 = (1/2)\,m\,2\,g\,|\Delta y| = mgh$. Therefore, $W_{net} = \Delta KE$

In the next chapter, we will define potential energy (PE). As the book falls, energy was transferred from the PE of the earth-book system into the KE of the book. The amount of energy transferred is $W_{grav} = mgh$.

Example: A book is lifted a height h by an external force (my hand) at constant velocity. Here, $F_{net} = 0$ (since constant velocity), so $W_{net} = 0$. The book does not speed up or slow down, so $\Delta KE = 0$. Hence, $W_{net} = \Delta KE$

The Work-KE Theorem provides a short-cut in some problems:

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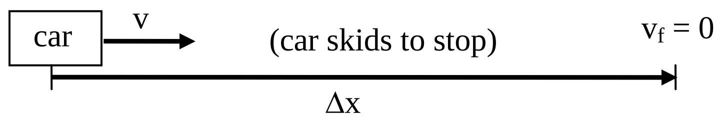
Example: A car of mass m is moving with speed v. The driver applies the brakes and the car skids to a stop. What was the magnitude of the work done by the friction force on the tires?



At first glance, it seems that we don't have enough info to answer the question. We don't know the coefficient of kinetic friction μ_K and we don't know how far (Δx) the car skidded. So how are we to compute the work done by friction $|W_{fric}| = F_{fric} \cdot |\Delta x| = \mu_K N \cdot |\Delta x|$?

Easy with the Work-KE Theorem: Here $F_{net} = F_{fric}$ so $|W_{fric}| = |W_{net}| = |\Delta KE| = (1/2)\, m\, v^2$.

Another question: If the initial speed v of the car is doubled, how much further does the car skid?

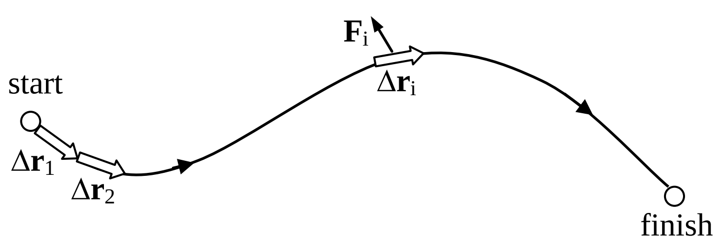
Answer: Begin with $|W_{fric}| = |W_{net}| = |\Delta KE| = (1/2)\, m\, v^2$

The work done is $|W_{fric}| = F_{fric} \cdot \Delta x = \mu_K N \cdot \Delta x = \mu_K mg \cdot \Delta x$. So we have …

$(1/2)\, m\, v^2 = \mu_K mg \cdot \Delta x$, $\Delta x = \dfrac{v^2}{2\, \mu_K\, g}$. Since $\Delta x \propto v^2$, if the v is doubled, the car skids 4 times as far. Notice that the distance that the car skids is independent of the mass of the car. This is a very useful fact in car crash investigations. Often, the investigator can estimate the speed of a car from the length of the skid mark, without needing to know anything about the car.

General definition of work.

Our earlier definition of work done by a force, $W_F \equiv \vec{F} \cdot \Delta \vec{r}$, only applies if the force is constant and the path is a straight line. What if the path is curvy and/or the force is not constant? Suppose the path is the curved line below and the force varies as the object moves.



To compute the work done, we break the path up into a large number of very small, straight-line segments, and label the segments with an index i. If the segment $\Delta\mathbf{r}_i$ is very small, it is essentially straight and the force $\mathbf{F}_i$ is constant over that segment, so the work done over the i^th^ segment is $\Delta W_i = \vec{F}_i \cdot \Delta \vec{r}_i$. The total work done over the whole path is $W = \sum_i \vec{F}_i \cdot \Delta \vec{r}_i$.

Taking the limit as the segment lengths become infinitesimal, the sum becomes an integral and the work done is

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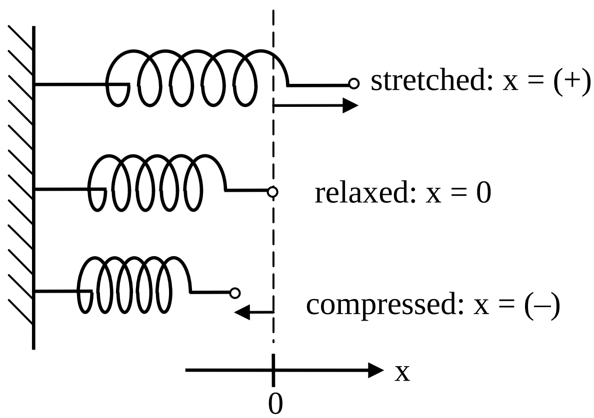
$$W_F = \int \vec{F} \cdot d\vec{r}$$

Don’t let the integral sign scare you. An integral is just a sum (the integral sign looks like an S for “sum”). The sum of little bits of work is the total work. An example of a force that varies with position is the force exerted by a stretched spring.

Springs
We want to derive an expression for the work done to stretch or compress a spring, so we take a little detour and talk about springs.

Most springs obey "Hooke's Law" which says that the force exerted by a spring is proportional to the displacement from the equilibrium (relaxed) position.



x is the amount that the spring is stretched (+) or compressed (–)

$|F_{spring}| \propto |x|$

double $|x|$ $\Rightarrow$ double $|F_{spring}|$

Hooke's “Law” : $\boxed{\mathbf{F_{spring} = -k\,x}}$

k = spring constant = measure of stiffness, big k $\Leftrightarrow$ stiff spring, small k $\Leftrightarrow$ floppy spring

units of k = [k] = [F]/[x] = N/m (newtons per meter)

Why the (–) sign in Hooke's Law? It's because the direction of the force exerted by the spring is opposite direction of displacement. When displacement is to the right (x +), the spring pulls back to the left (F –); when x is (–), F is (+).

Why Hooke's "Law" in quotes? Because it is not really a law. It is only approximately true for most springs as long as the extension is not too great. If a spring is stretched past its "elastic limit", the spring will permanently deform, and it will not obey Hooke's Law.

Incidentally, Robert Hooke (1635-1703) was a brilliant English scientist, a contemporary of Newton’s, and Newton *hated* him. On about two occasions, Hooke figured out something at about the same time or earlier than Newton. One example is “Hooke’s Law”. Newton was very insecure, intellectually, and he got furious when anyone figured out something before he did.

We now show that the work done by an external force F_{ext} (such as the force from my hand) to stretch or compress a spring by an amount x is given by

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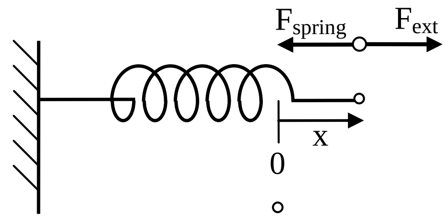
$$W_{ext} = \frac{1}{2} k x^2$$

To hold the spring stretched a displacement x, I have to exert an external force $F_{ext} = +k\,x$. To slowly stretch the spring from $x_i = 0$ to x_f , I have to apply a force that increases from zero to kx_f. Our general definition of work in 1D is



$$W_F = \int F\,dx$$.

$$W_{ext} = \int_{xi}^{xf} F_{ext}\,dx = \int_0^x k\,x\,dx = \tfrac{1}{2} k\,x^2 \Big|_0^x = \tfrac{1}{2} k\,x^2$$ Done.

If you don't yet know how to do integrals, here's another way of solving for the work. As the spring is stretched, the force varies linearly from zero (unstretched) to +kx (fully stretched). The *average* force applied is $\tfrac{1}{2} k\,x$. We can pretend the force is constant, equal to the average force, and the work done is $W_{ext} = \text{force} \times \text{distance} = \tfrac{1}{2} k\,x \cdot x = \tfrac{1}{2} k\,x^2$.

Notice that the work done by the external force is always positive, regardless of whether the spring is stretched (x positive) or compressed (x negative).

Appendix.

Derivation of Work-Kinetic Energy Theorem in 1D:

This derivation requires some knowledge of integrals.

Working in 1D, I can drop the arrows over the vectors, since direction can be represented by a sign in 1D. Starting with the general definition of work done by a force and using $F_{net} = ma$ and $a = dv/dt$, we have

$$W_{net} = \int F_{net}\,dx = \int m\,a\,dx = \int m \frac{dv}{dt}\,dx$$

The velocity v can be regarded as a function of x, and x is a function of t: v = v[x(t)]. By the chain rule, $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$. Multiplying both sides by dx gives $\frac{dv}{dt}dx = \frac{dx}{dt}dv$, so we have

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$$W_{net} = \int m \frac{dv}{dt}\, dx = \int m \frac{dx}{dt}\, dv$$

$$= \int m\, v\, dv = \frac{1}{2} m v^2 \Big|_{v1}^{v2} = \frac{1}{2} m v_2^{\,2} - \frac{1}{2} m v_1^{\,2} = \Delta KE$$

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