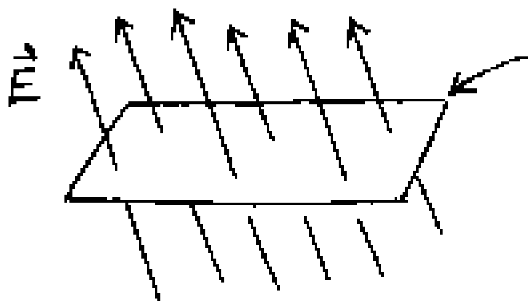
Knight Ch 27, $\boxed{\text{Gauss's Law}}$

New concept: electric flux Φ thru a surface

(begin w/ flat surfaces, uniform $\vec{E}$)

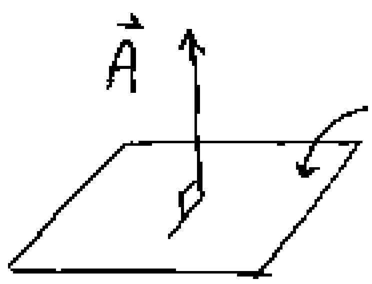


surface w/ area A has some electric flux thru it.

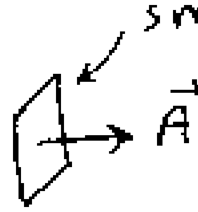
Define <u>surface vector</u> $\vec{A} = A\,\hat{n}$

Magnitude $A = |\vec{A}|$ = area A of surface

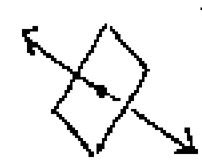
direction of $\vec{A}$ = direction $\perp$ to surface
= direction of unit normal $\hat{n}$



area A



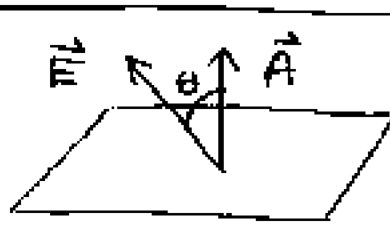
smaller area, shorter $\vec{A}$

(Note there are <u>2</u> possible directions! )

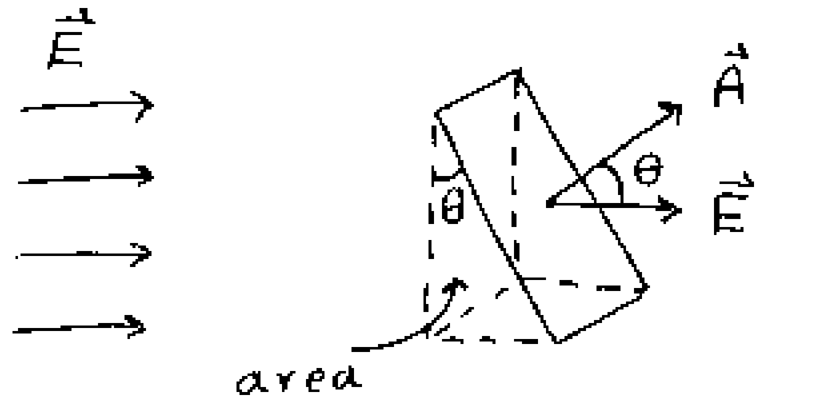
Flux Φ thru surface $\vec{A}$ defined as

$$\boxed{\Phi = \vec{E}\cdot\vec{A} = E\,A\cos\theta}$$

← for special case $\vec{E}$ = const, surface <u>flat</u>



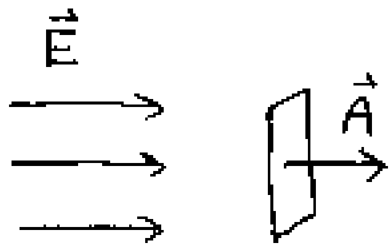
$A\cos\theta$ = area <u>facing</u> $\vec{E}$



Φ large only if $A\cos\theta$ is large

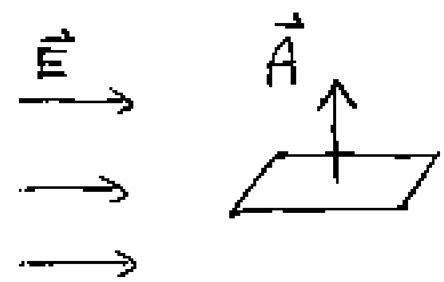
$A\cos\theta$ = projection of area $\perp$ to $\vec{E}$

Think of field lines as rain flowing thru area $\vec{A}$. Flux = amount of rain thru area



$\theta = 0$, $A\cos\theta = A$

$\Rightarrow \Phi$ <u>max</u>



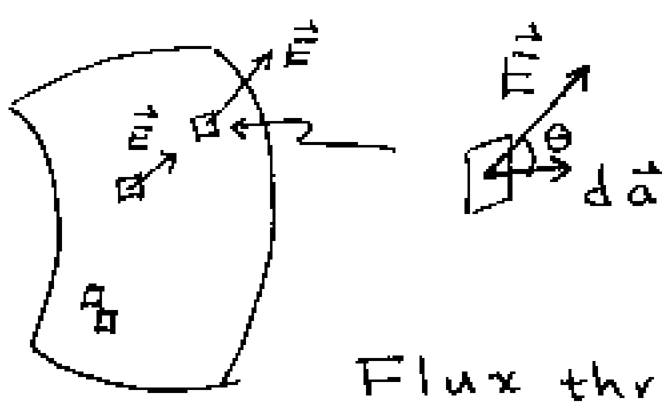
$\theta = 90^\circ$ $\cos\theta = 0$

$\Phi = 0$

Flux thru surface $\propto$ # field lines crossing surface

If $\vec{E}$ not const or area (surface) curved,

$\Phi = \vec{E}\cdot\vec{A} \longrightarrow \Phi = \int \vec{E}\cdot d\vec{a}$ = "surface integral"



Break surface up into many tiny segments of area da

Flux thru segment $i = \Phi_i = \vec{E}_i \cdot d\vec{a}_i$

Total flux thru whole surface =

$$\Phi = \sum_i \Phi_i = \sum_i \vec{E}_i \cdot d\vec{a}_i \longrightarrow \int \vec{E} \cdot d\vec{a}$$

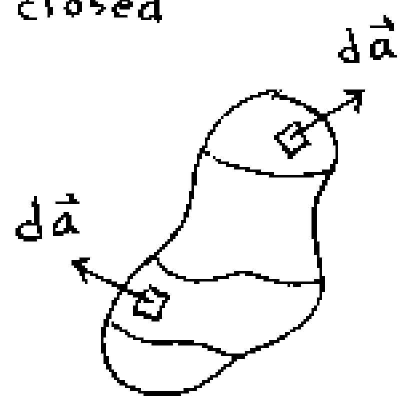
$\boxed{\text{Gauss's Law}}$ (the 1st of 4 Maxwell's Eq'ns)

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{in}}{\varepsilon_0}$$

The electric flux thru any closed surface S is a constant ($1/\varepsilon_0$) times the net charge Q enclosed by S.

$\oint \vec{E} \cdot d\vec{a}$ = "closed" surface integral

"closed"



← for closed surface,

direction of $d\vec{a}$ = outward normal

ε_0 related to K by $K = \frac{1}{4\pi\varepsilon_0}$

$\varepsilon_0 = 8.85 \times 10^{-12}$ (SI units)

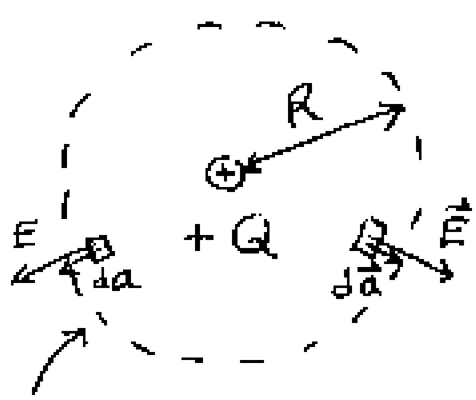
$$F_{coul} = \frac{K|q_1 q_2|}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r^2}$$

Gauss's Law can be derived from Coulomb's Law *if* charges are stationary, but Gauss's Law is more general than Coulomb's Law. Coulomb only true if q's are stationary. Gauss true *always* whether or not q's are moving.

Easy to show Gauss's Law is *consistent* w/ Coulomb's Law. Coulomb says...

$$E = \frac{kQ}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \quad \text{(for } \underline{\text{point}} \text{ charge } Q\text{)}$$

Does Gauss agree?



imaginary surface S
sphere, radius R

$$\oint_S \vec{E} \cdot d\vec{a} = \oint E\, da$$

(since $\vec{E} \parallel d\vec{a}$)

$$= E \oint da \quad \text{(since } E \text{ const)}$$

$$= E \cdot A = E\, 4\pi R^2$$

$$= \frac{Q}{\varepsilon_0} \quad \text{(says Gauss)}$$

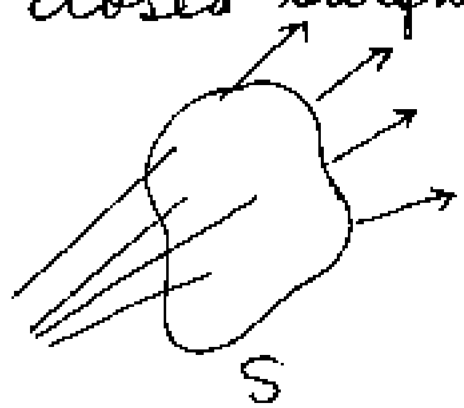
$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_0 R^2} \quad \checkmark$$

For <u>closed</u> surface ($d\vec{a}$ <u>outward</u>), flux is (+) over parts of surface where $\vec{E}$-field lines are <u>exiting</u>; flux is (−) where field lines are <u>entering</u>

The total (net) flux thru a closed surface

$$\Phi \propto \left[(\#\text{ lines exiting}) - (\#\text{ lines entering}) \right]$$

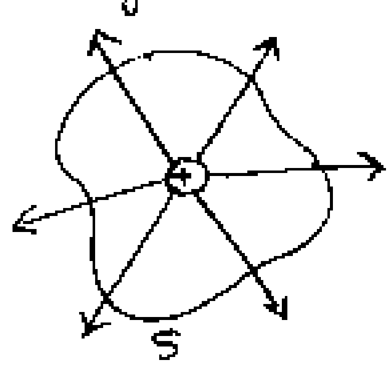
If field lines come from charge exterior to closed surface, then (# lines out = # lines in)



$\Rightarrow$ net flux = 0

$$\oint_S \vec{E} \cdot d\vec{a} = 0$$

If field lines come from charge inside closed surface, then <u>all</u> field lines from charge exit surface

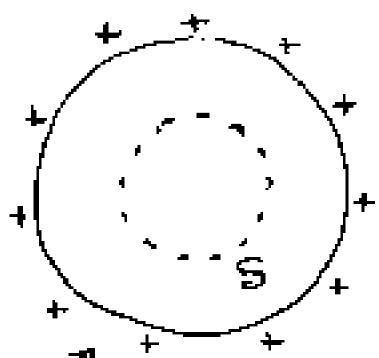


Total flux $\propto$
field lines originating inside
$\propto$ charge inside

$$\oint \vec{E} \cdot d\vec{a} = Q_{in}/\varepsilon_0$$

In situations w/ high symmetry Gauss's Law allows quick calculation of $\vec{E}$-field. Only easy to compute $\oint \vec{E}\cdot d\vec{a}$ if symmetric situation

Example: $\vec{E}$ inside uniform spherical shell of charge



uniform charge/area σ on surface; no Q inside

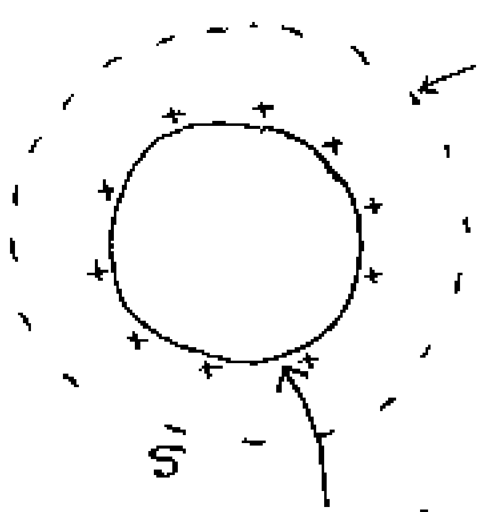
By symmetry, $\vec{E}$ must be radial (along a radius)

$\Rightarrow \vec{E} = E(r)\,\hat{r}$

$\Rightarrow \vec{E}\cdot d\vec{a} = E\,da$ (since $\vec{E} \parallel d\vec{a}$)

$$\oint_S \vec{E}\cdot d\vec{a} = \oint E\,da = E\oint da = EA = \frac{Q_{in}}{\varepsilon_0} = 0$$

$\Rightarrow \vec{E} = 0$ everywhere inside shell



total charge Q

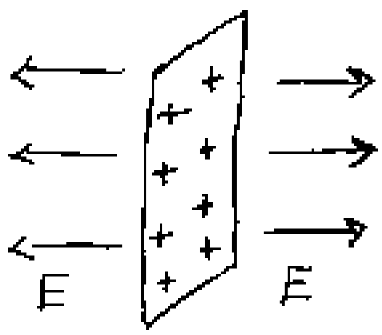
With gaussian surface S' outside shell, can show

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r}$$ everywhere outside shell

$\vec{E}$-field near infinite, uniform plane of charge

$\sigma = \frac{Q}{A}$ = charge per area of plane

"sigma" for "surface" (Knight uses η)



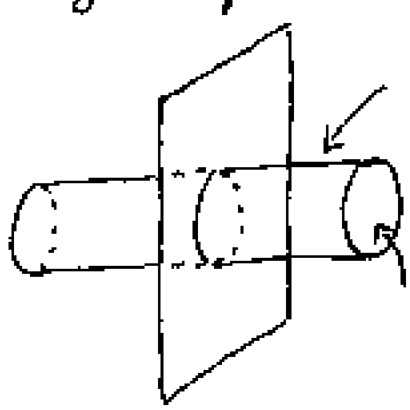
$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

independent of distance from plane!

$\vec{E} \perp$ plane

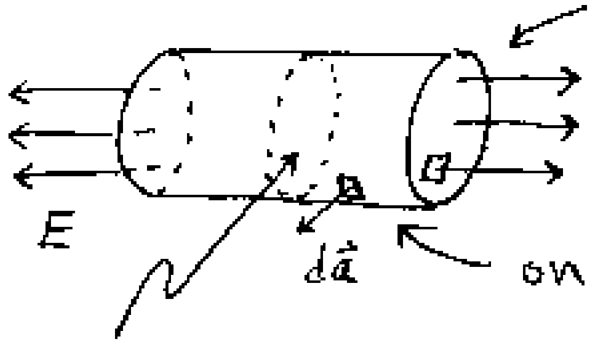
Proof: Apply Gauss's Law:

By symmetry, must have $\vec{E} \perp$ plane



imaginary gaussian surface S = cylinder

area A of endcap



on endcaps, $\vec{E} \cdot d\vec{a} = E\,da$ ($\vec{E} \parallel da$)

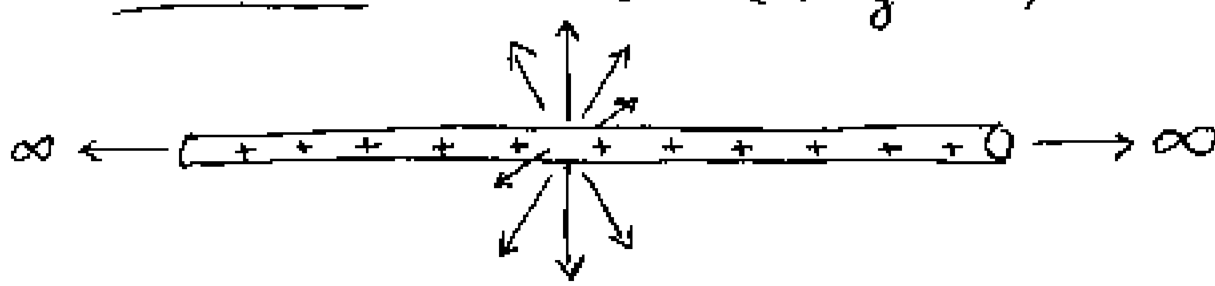
on sides, $\vec{E} \cdot d\vec{a} = 0$ ($\vec{E} \perp d\vec{a}$)

$Q_{in} = \sigma \cdot A$ $\qquad \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{in}}{\varepsilon_0}$

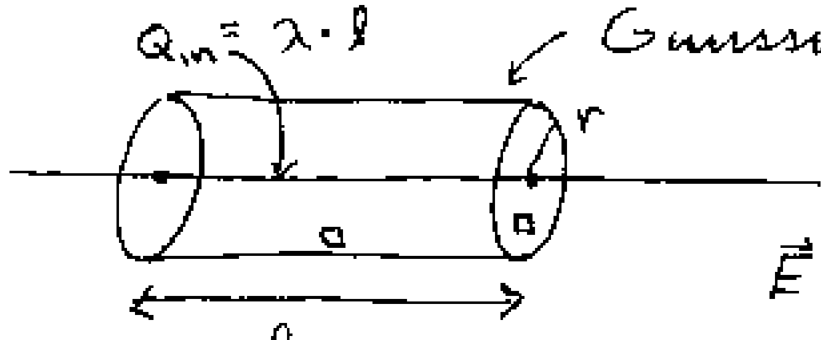
$$\oint_S = \int_{left} + \int_{side} + \int_{right} = EA + 0 + EA$$

$$\Rightarrow \quad 2E\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{2\varepsilon_0} \quad \checkmark$$

<u>Infinite</u> line of charge w/ charge/length $= \lambda$



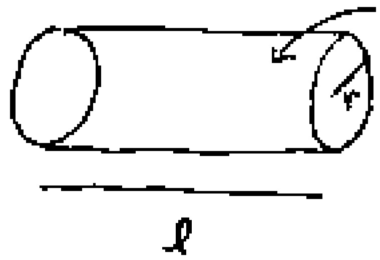
By symmetry, $\vec{E} \perp$ line $\Rightarrow E = E(r)$
$\vec{E}$ in <u>radial</u> direction, depends only on $r =$ radial distance



$\vec{E} \cdot d\vec{a} = 0$ on flat endcaps

$\vec{E} \cdot d\vec{a} = E\,da$ on curved side

$$\oint_S \vec{E} \cdot da = \int_{\substack{\text{curved}\\ \text{side}}} \vec{E} \cdot d\vec{a} = \int_{\substack{\text{curved}\\ \text{side}}} E\,da = E \int da = E A_{side}$$



area of side $= 2\pi r \ell$

$$\oint \vec{E} \cdot d\vec{a} = Q_{in}/\varepsilon_0 \Rightarrow$$

$$E\,(2\pi r \cancel{\ell}) = \frac{\lambda \cdot \cancel{\ell}}{\varepsilon_0}, \quad \boxed{E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

Conductors in electrostatic equilibrium.

- $\vec{E} = 0$ inside a conductor (in equilibrium)

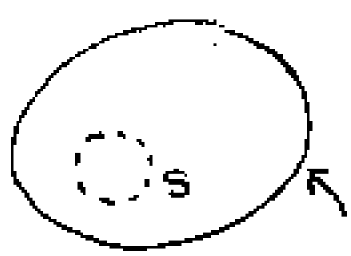
  proof: If $\vec{E} \neq 0$ inside, then conduction e's feel a force $\vec{F} = q\vec{E}$ and will move.

  Moving charge $\Longleftrightarrow$ not electrostat. equil.

  In and on a conductor in equil., the charges arrange themselves so that $\vec{E} = 0$ everywhere inside (otherwise, not yet in equil.)

- The interior of a conductor in equil. can have no net charge (electrons + protons must have equal density).

  proof from Gauss's Law: consider any closed surface, S, within conductor



conductor

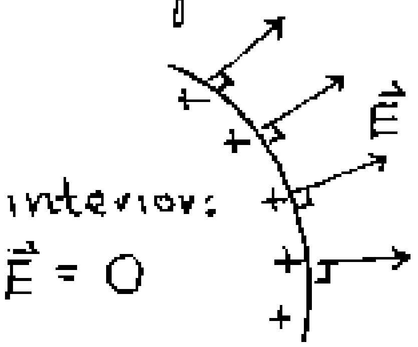
$$\oint_S \vec{E} \cdot d\vec{a} = 0 \quad (\text{since } \vec{E} = 0 \text{ on } S)$$

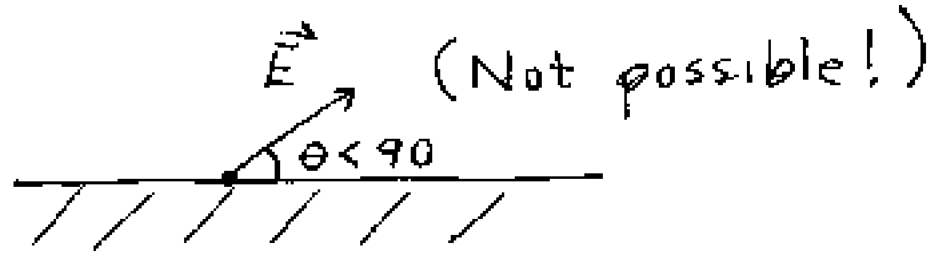
$$\Rightarrow Q_{in} = 0 \ \checkmark$$

- If a conductor has a net charge Q, the charge must reside on the surface
  - just showed it can't reside in interior $\Rightarrow$ must be on surface

- $\vec{E}$-field near (at) surface of charged conductor must be $\perp$

$\vec{E}$-field near charged conductor must be $\perp$ to surface.

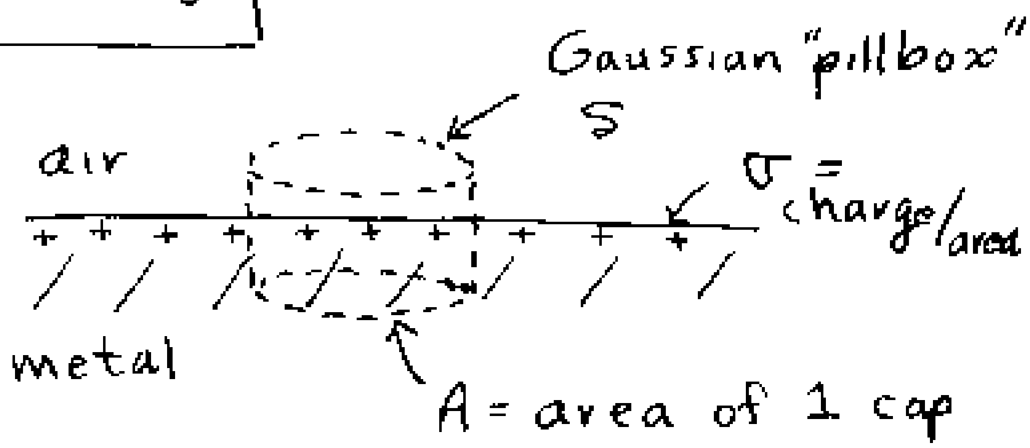


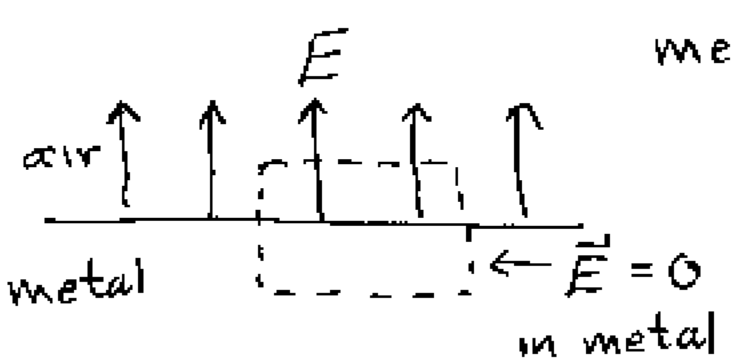


If $\vec{E}$ not $\perp$ to surface, there is a component of $\vec{E}$ along surface ($E_x = E\cos\theta$), which would push conduction e's along surface
$F_x = q E_x \Rightarrow$ would not have equilibrium.

- $\vec{E}$-field near surface of charged conductor has magnitude $\boxed{E = \sigma/\varepsilon_0}$

Proof by Gauss:





$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{in}}{\varepsilon_0}$$

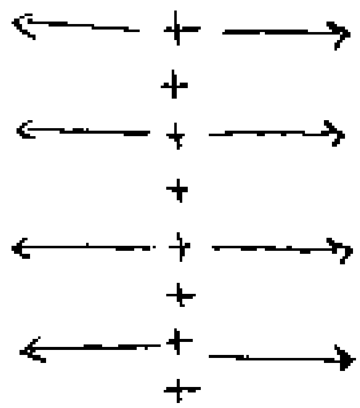
$$E \cancel{A} = \sigma \cancel{A} / \varepsilon_0$$

$$\Rightarrow E = \sigma/\varepsilon_0$$

(Notice similar, but different, from formula for $\vec{E}$ due to plane of charge $\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$)

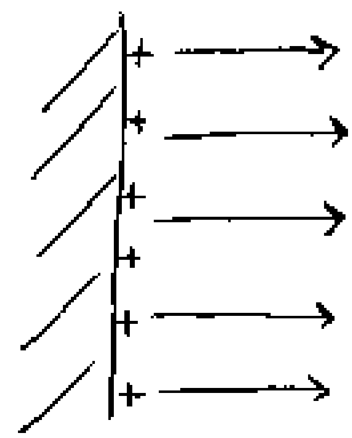
A puzzling situation has arisen!

Proved before that plane of charge creates field $E = \sigma/2\varepsilon_0$. Now have proved that plane of charge on surface of metal makes $E = \sigma/\varepsilon_0$



$E = \dfrac{\sigma}{2\varepsilon_0}$

conflict?

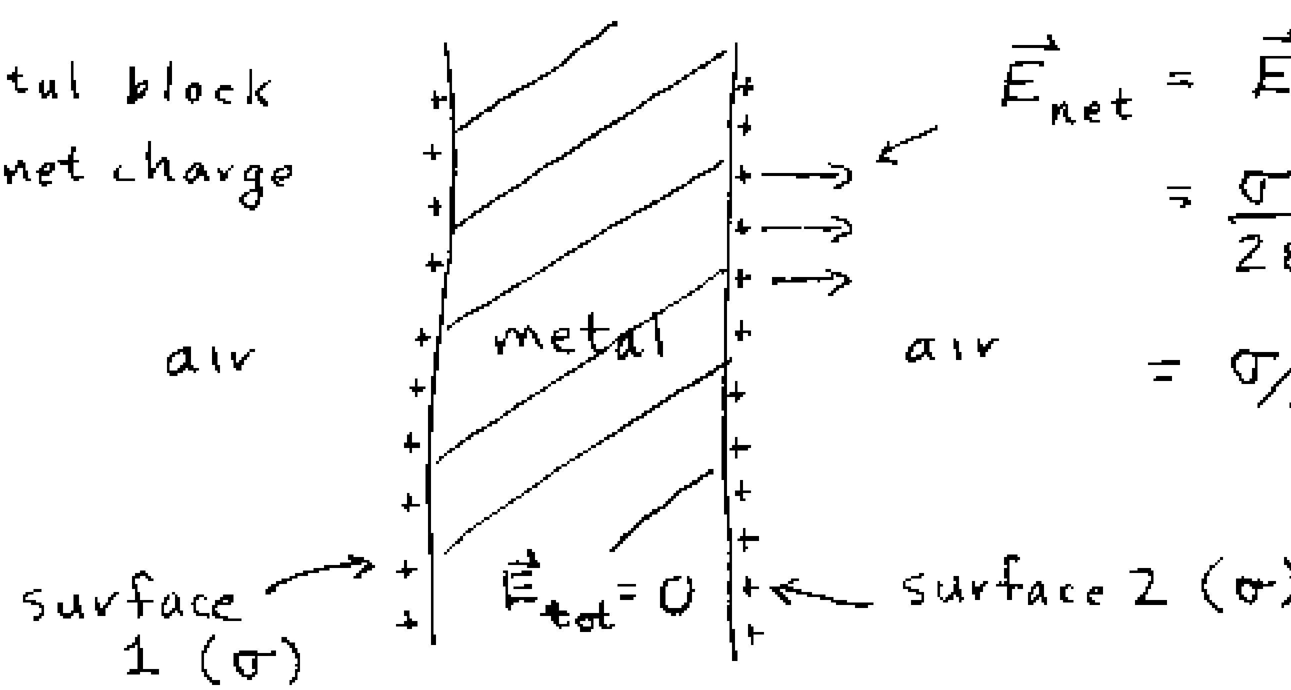


$E = \dfrac{\sigma}{\varepsilon_0}$

How can both be correct?

The $\vec{E}$-field anywhere is always due to all charges everywhere, including charges on far-away surfaces. Field near surface of metal is not only due to charges on nearby surface.

Metal block
w/ net charge



$$\vec{E}_{net} = \vec{E}_1 + \vec{E}_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \sigma/\varepsilon_0 \checkmark$$