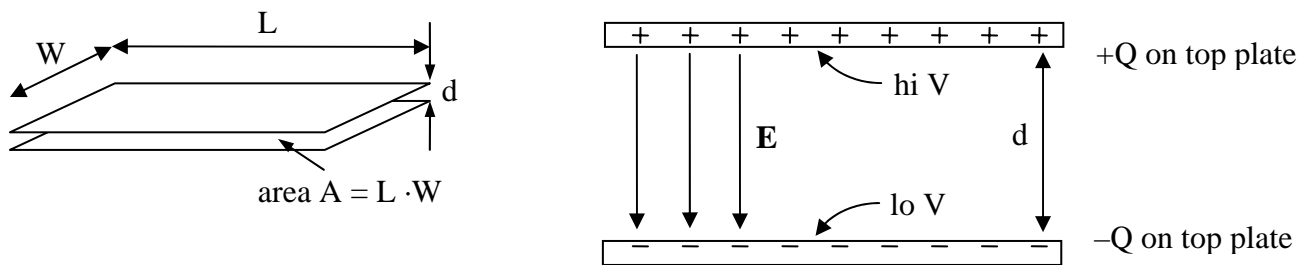


Capacitors

A **capacitor** is simply two pieces of metal near each other, separated by an insulator or air. A capacitor is used to store charge and energy.

A **parallel-plate capacitor** consists of two parallel plates separated by a distance d , each plate with area A . If A is large and d is small, the plates are effectively infinite planes, and the E-field is uniform and entirely in-between the plates.



Charges are always on the *inside* surfaces, because (+) attracts (-). The outside surfaces remain uncharged.

"Charge Q on a capacitor" always means $+Q$ on one plate, $-Q$ on the other plate. Capacitors are charged by transferring (-) charge from one plate to the other. Taking (-) charge off a plate leaves behind an equal-sized (+) charge.

The charges make an E-field, which means a voltage difference between the plates. The "voltage V on a capacitor" always means the voltage difference ΔV between the plates.

$$|\Delta V| = E \cdot d = V \Rightarrow V \propto E \propto Q \Rightarrow \text{ratio } \frac{Q}{V} = \text{constant}$$

It is always true that $\Delta V \propto E$, since $\Delta V = -\int_i^f \vec{E} \cdot d\vec{r}$ (double the E-field everywhere and ΔV

doubles.) And it is always true the $E \propto Q$, since $\vec{E} = \sum_i k \frac{q_i}{r_i^2} \hat{r}_i$ (double all the charges

everywhere and E doubles). So the ratio Q/V is always a constant: if you double the charge Q , the V ($=\Delta V$) is guaranteed to double.

Definition: **capacitance** C of a capacitor: $C = \frac{Q}{V}$

If we double the charge Q , the voltage V doubles, but the ratio Q/V remains constant.

[Remember: Q means $+Q$ and $-Q$, V means ΔV .]

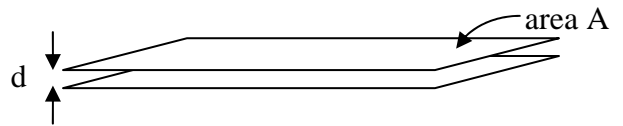
units $[C] = \text{coulomb} / \text{volt} = \text{farad (F)}$

Big capacitance (1F) \Rightarrow can store a big Q with a small V

Small capacitance (nF = 10^{-9} F) \Rightarrow small Q stored with a big V

For a parallel-plate capacitor, with air or vacuum between the plates, the capacitance is

$$C = \frac{\epsilon_0 A}{d} \quad (\text{air or vacuum separating plates})$$



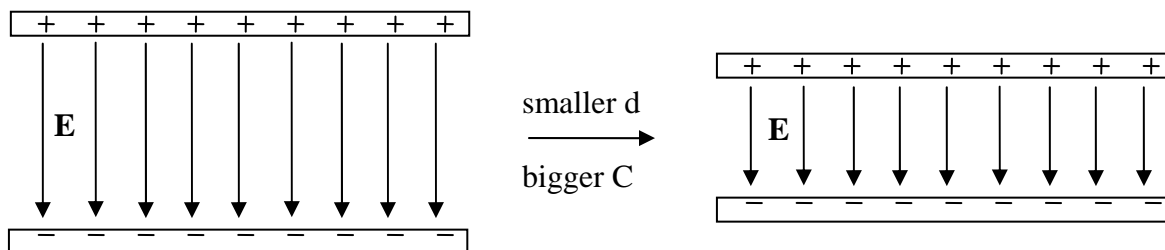
ϵ_0 ("epsilon-naught") is the same constant that appeared in Gauss's Law.

Proof: $\Delta V = V = E \cdot d = \frac{\sigma}{\epsilon_0} \cdot d = \frac{Q}{A} \cdot \frac{d}{\epsilon_0}$ (We have used $E = \frac{\sigma}{\epsilon_0}$ for a capacitor.)

Rearranging, we get $C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$. Done.

Notice that the capacitance of a parallel-plate capacitor depends only on the size and shape of the two metal parts. This turns out to be true of all capacitors. The capacitance of two pieces of metal depends solely on their geometry.

Note that this formula means C increases as d decreases. Why? If Q is kept fixed, we have the same magnitude E -field (because same charge density $\sigma = Q/A$ creates the $E = \sigma/\epsilon_0$). Smaller d and same-sized E means smaller voltage $V = E d$. Same Q and smaller V means bigger $C = Q/V$.



A farad is a *huge* capacitance. For example, suppose we make a parallel plate capacitor with area $A = 1 \text{ m}^2$ (big) and separation $d = 1 \text{ mm} = 0.001 \text{ m}$. The capacitance is only

$$C = \frac{\epsilon_0 A}{d} = \frac{(8.85 \times 10^{-12})(1)}{10^{-3}} \simeq 9 \times 10^{-9} \text{ F} = 9 \text{ nF} \quad (\text{tiny!})$$

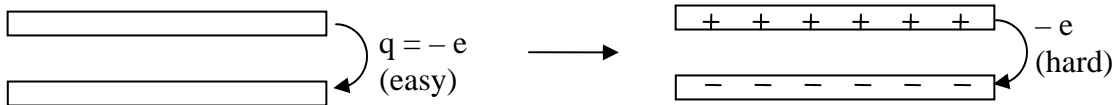
Multi-farad capacitors in small packages are made by making d very small.

$d \approx$ atomic dimensions \leq nm (nanometer) is possible.

Stored Energy in Capacitors

It takes *work* to charge a capacitor, because it is difficult to transfer more electrons from the (+) plate to the (-) plate. The work required to transfer a charge q across a voltage difference " V " = ΔV is $\Delta \text{PE} = q \Delta V$.

When we charge up a capacitor from $q_{\text{initial}} = 0$ to $q_{\text{final}} = Q$, we transfer electrons one at a time. The first electron is easy to transfer since $V = \Delta V = 0$ initially, but the later electrons take more and more work to transfer as Q (and ΔV) builds up.



Total work to charge capacitor = electrostatic potential energy stored in capacitor =

$$U = \frac{1}{2} Q V$$

(We use U for energy to avoid confusion with E for electric field.)

Why the $(1/2)$? Why not $\Delta \text{PE} = W_{\text{ext}} = Q V$? While transferring the total charge Q , the voltage difference increased from 0 to V . The *average* value was $(1/2)V$.

We can show this more rigorously by doing an integral. When the voltage difference between the plates is V , the work required to transfer an extra bit of charge dq is $dU = V dq = (q/C) dq$.

The total work (= total PE) to charge the capacitors is the sum (the integral) of the works done to

$$\text{transfer all the bits of charge: } U = \int dU = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}$$

Can rewrite U in various ways using $C = Q/V$, $Q = C V$, $V = Q/C$:

$$U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

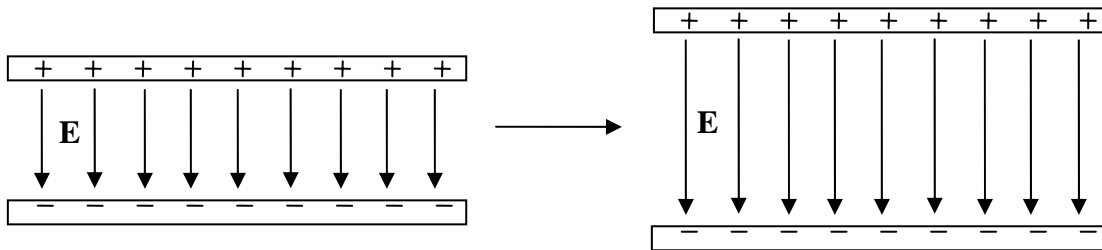
Where is this energy? The E-field contains the energy. It takes work to create an E-field. It turns out that the energy *per volume* (the energy density) of the E-field is given by

$$u = \frac{U}{\text{Vol.}} = \frac{1}{2}\epsilon_0 E^2$$

"Proof": $U = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \frac{1}{2}\epsilon_0 E^2 \underbrace{(A \cdot d)}_{\text{volume}} \Rightarrow u = \frac{U}{\text{Vol.}} = \frac{1}{2}\epsilon_0 E^2$

(This is a proof for the special case of a parallel-plate capacitor only, but the result turns out to be true always.)

The energy $U = (1/2)QV$ of a charged capacitor is in the E-field between the plates. If we pull the plates apart, keeping the charge Q fixed, we increase the volume which contains E-field and the total energy increases. It was *hard* to pull the plates apart, because opposite charges attract.



The work we did went into creating more E-field (same size E-field over larger volume).

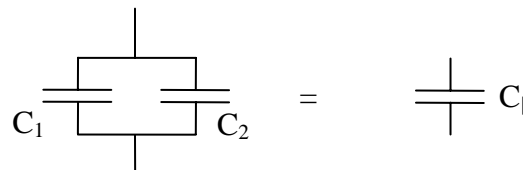
It turns out that the work done to assemble a collection of charges ($W_{\text{ext}} = \Delta U = q \Delta V$) is equal to

the energy in the E-field created: $U = \int \left(\frac{1}{2}\epsilon_0 E^2\right) dV$ (volume integral)

Capacitors in parallel or in series.

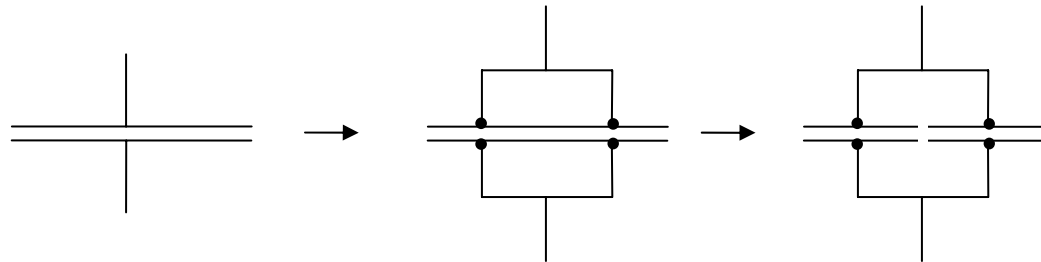
Symbol for capacitor :

For capacitors in parallel, $C_{\parallel} = C_1 + C_2$

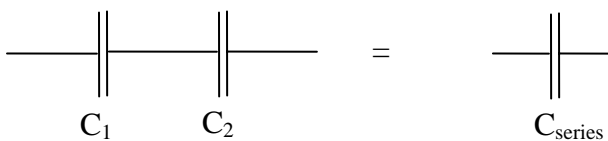


A big C is equivalent to two smaller C side by side:

"Proof" :

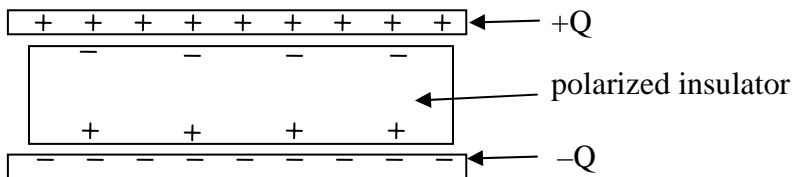


For capacitors in series:
$$C_{\text{series}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$



Capacitors filled with dielectrics

The capacitance C of a capacitor can be increased by placing an insulator ("dielectric") between plates. The dielectric is polarized by the charges on the plates.



For fixed Q on the plates, the E -field between the plates is reduced when a dielectric is inserted because the polarization charge on the dielectric partially cancels the charge Q on the plates.

smaller $E \Rightarrow$ smaller $V = |\Delta V| = E d$, smaller V and same Q on plates \Rightarrow larger $C = Q / V$

Let's call the original E -field E_0 and the final, smaller, E -field E . The original E -field between the plates has been reduced by a dimensionless factor called K ($K =$ dielectric constant):

$$K = \frac{E_0}{E}. \text{ The dielectric constant } K \text{ is greater than } 1 \text{ and the value depends on the material the insulator is made of.}$$

For a capacitor filled with a dielectric:
$$C = \frac{\epsilon_0 K A}{d}$$