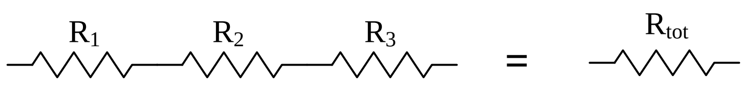
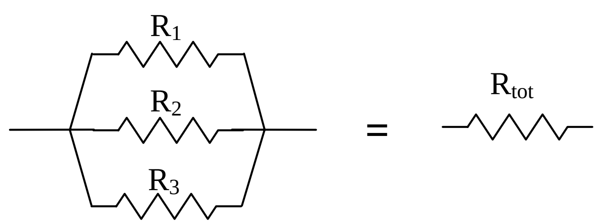
More Circuits

In an electrical circuit, circuit elements such as resistors and batteries can be connected together *in series* or *in parallel*. Resistors in series are connected like links in a chain; resistors in parallel are side-by-side, like so:

in **series:**



in **parallel:**



Series: $R_{tot} = R_1 + R_2 + R_3$, $R_{tot} > R_1, R_2, R_3$

Parallel: $R_{tot} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$, $R_{tot} < R_1, R_2, R_3$

Resistors in series act like a single large resistor.

Resistors in parallel act like a single small resistor.

Proof:

Resistors in Series: $I_{tot} = I_1 = I_2$, $\Delta V_{tot} = \Delta V_1 + \Delta V_2 \Rightarrow$

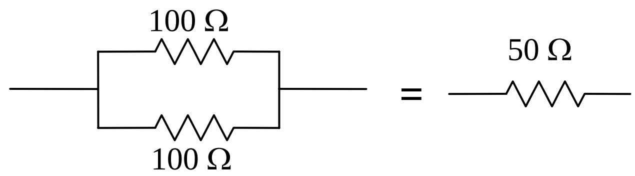
$$R_{tot} = \frac{\Delta V_{tot}}{I_{tot}} = \frac{\Delta V_1 + \Delta V_2}{I_{tot}} = \frac{\Delta V_1}{I_1} + \frac{\Delta V_2}{I_2} = R_1 + R_2$$

Resistors in Parallel: $I_{tot} = I_1 + I_2$, $\Delta V_{tot} = \Delta V_1 = \Delta V_2 \Rightarrow$

$$\frac{I_{tot}}{\Delta V_{tot}} = \frac{I_1 + I_2}{\Delta V_{tot}} = \frac{I_1}{\Delta V_1} + \frac{I_2}{\Delta V_2} \Rightarrow \frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$$

Examples of parallel resistors:

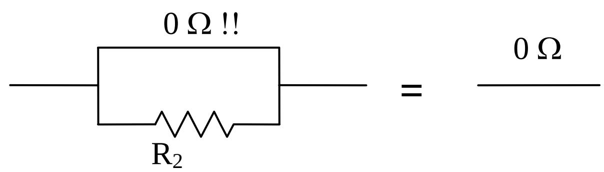
1) Two 100 Ω resistors in parallel:



$$R_{tot} = \frac{1}{\frac{1}{100\,\Omega}+\frac{1}{100\,\Omega}} = \frac{1}{\left(\frac{2}{100}\right)} = \frac{100}{2} = 50\,\Omega$$

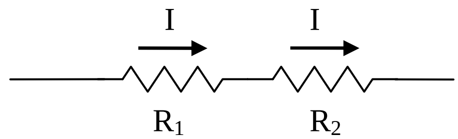
2) 10 Ω in parallel with 0 Ω wire:

$$R_{tot} = \frac{1}{\frac{1}{0}+\frac{1}{10}} = \frac{1}{\infty} = 0$$

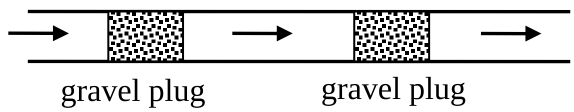


Key points:

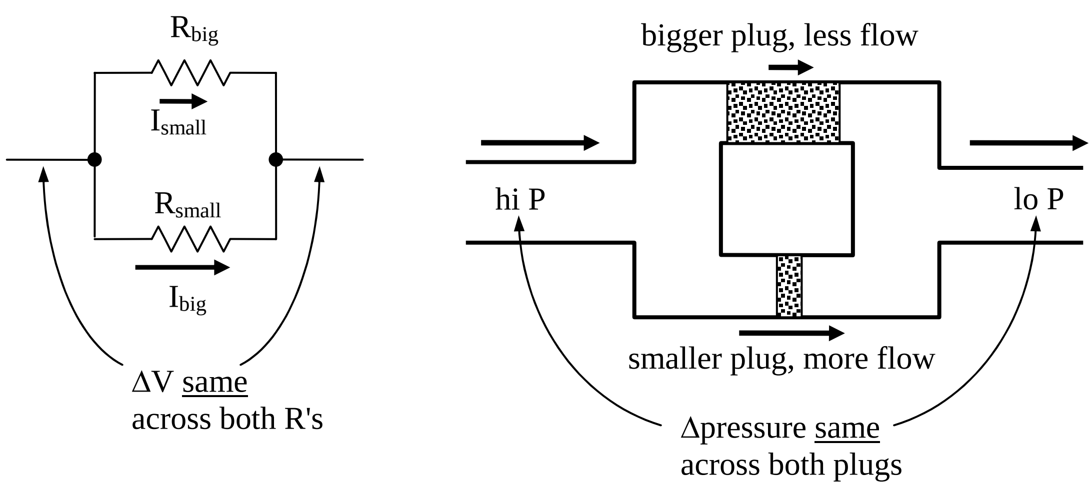
- The <u>current</u> is the <u>same</u> for resistors <u>in series</u>. Current is not "used up".



Think of the water pipe analogy: two gravel plugs in series, same flow (same gal/min) through both plugs (assuming no leaks or bubble in the pipe)



- Adding another resistor in series always increases the total resistance.
- The <u>voltage</u> difference across each resistor is the <u>same</u> for resistors <u>in parallel</u>.

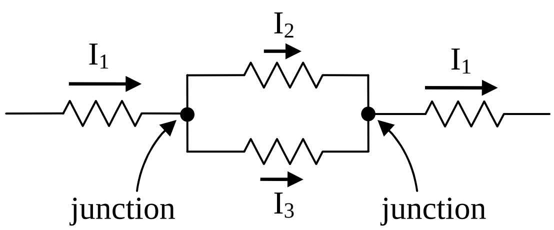


Both resistors in parallel have the same $\Delta V = V = I_{big} R_{small} = I_{small} R_{big}$.

- Adding another R in parallel always decreases the total resistance. Like adding another pipe along side the original pipe $\Rightarrow$ allows more flow $\Rightarrow$ smaller total resistance

Kirchhoff's two rules for analyzing circuits (*Kirchhoff* is really spelled that way: 2 h's, 2 f 's)

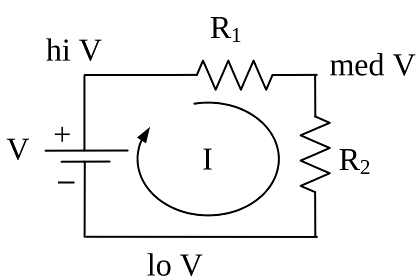
Kirchhoff's Current Rule (also called the Junction Rule)
The total current into any junction = total current out
(*junction* = place where 3 or more wires meet)

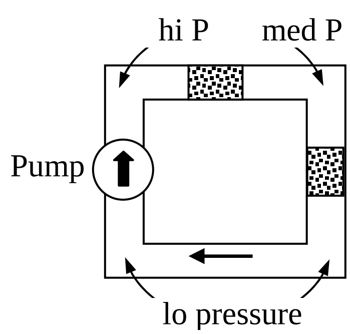


$$I_1 = I_2 + I_3$$

This is also called Conservation of Current. In steady-state, the charge is not building up anywhere, it is just flowing along at a steady rate. So the current into any portion of the circuit must equal the current coming out of that portion, otherwise charge would be building up in that part of the circuit.

Kirchhoff's Voltage Rule (also called the Loop Rule)
The sum of the voltage rises around any complete loop in a circuit = sum of the voltage drops around the same loop. Voltage rises and drops must sum to zero, since we must return to the same voltage after one complete loop.

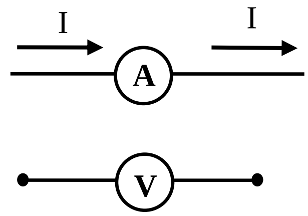




$$\underbrace{V}_{\text{rise}} = \underbrace{IR_1}_{\text{fall}} + \underbrace{IR_2}_{\text{fall}} = I(R_1 + R_2) \Rightarrow I = \frac{V}{R_1 + R_2}$$

Remember: voltage is a kind of "electrical pressure" or "electrical height" . If you go around a complete circuit and return to the same place, you are back at the same pressure (or height). So rises must equal drops.

Ammeters and Voltmeters

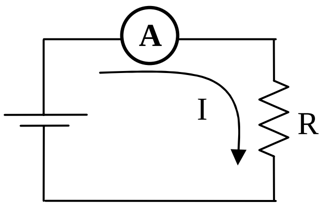


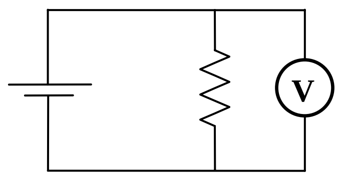
An ammeter measures the current through itself

A voltmeter measures the voltage difference between its terminals.

To measure the current through a resistor R, must place the ammeter <u>in series</u> with R.

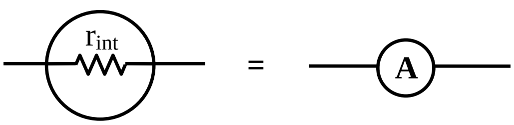
To measure the voltage across R, must place voltmeter <u>in parallel</u> with R.



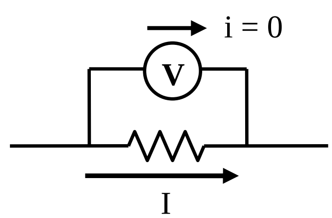


An ideal ammeter has zero internal resistance $r_{internal} = 0$, so current I is not affected.

An ideal voltmeter has $r_{internal} = \infty$, so no current flows through $\Rightarrow$ currents and voltages in rest of circuit are not affected.



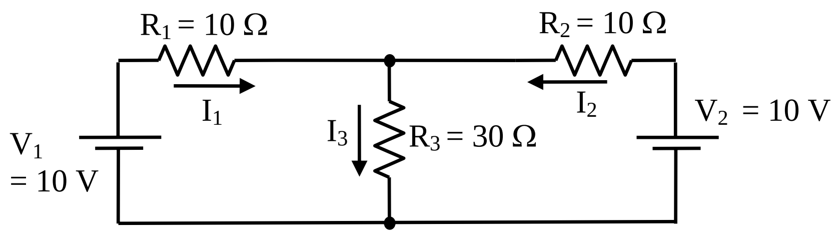
ideal ammeter: $r_{int} = 0$



ideal voltmeter: $r_{int} = \infty$

Circuits with multiple loops and batteries

Have a circuit with known V's and known R's. Seek the I's.



Procedure:

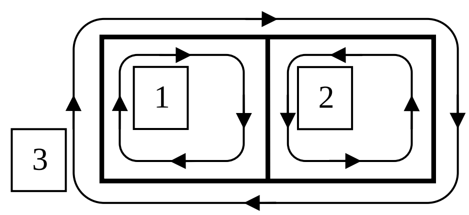
I. Guess direction of I through each R. Draw I arrows, label each (I_1 thru R_1, etc). (Directions of currents not always obvious, so just guess. If you guess wrong, value of current I will come out with a negative value.)

II. K's Current Law gives 1 or more equations:

(eq'n 1) $I_1 + I_2 = I_3$ [3 unknowns (I_1 , I_2 , I_3) $\Rightarrow$ will need 3 eq'ns to solve]

III. K's Voltage Law gives an equation for each complete loop in the circuit.

3 loops in this circuit. Only need 2 more equations, so only 2 of the 3 loop equations are needed.



Loop 1: $V_1 = I_1 R_1 + I_3 R_3$ (eq'n 2)

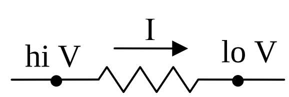
Loop 2: $V_2 = I_2 R_2 + I_3 R_3$ (eq'n 3)

Loop 3: $\underbrace{+V_1}_{\text{rise}} \ \underbrace{-\ I_1 R_1}_{\text{drop}} \ \underbrace{+\ I_2 R_2}_{\text{rise}} \ \underbrace{-\ V_2}_{\text{drop}} = 0$ (Moving CW around loop 3)

Don't need the equation from loop 3, because already have 3 equations.

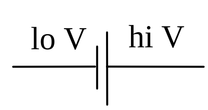
Remember! In a resistor,

if we move in the direction of the current, V drops, ΔV is negative ;



if we move in the direction opposite current, V rises, ΔV is positive.

In a battery,



if move from (–) to (+) terminal, V rises ;

if move from (+) to (–) terminal, V drops.

We now have 3 equations in 3 unknowns (I_1 , I_2 , I_3) :

(1) $I_1 + I_2 = I_3$

(2) $V_1 = I_1 R_1 + I_3 R_3$

(3) $V_2 = I_2 R_2 + I_3 R_3$

The physics part of this problem is over; now we have a messy algebra problem.

How do we solve?

Eqn (1) says we can substitute ($I_1 + I_2$) for I_3 . $\Rightarrow$ Eliminate I_3 in equations (2), (3):

$V_1 = I_1 R_1 + (I_1 + I_2) R_3$ $\qquad$ $V_2 = I_2 R_2 + (I_1 + I_2) R_3$

Rearrange:

A. $V_1 = I_1 (R_1 + R_3) + I_2 R_3$ $\qquad$ **B.** $V_2 = I_1 R_3 + I_2 (R_2 + R_3)$

Now have 2 equations (A, B) in 2 unknowns (I_1, I_2). Now combine these to eliminate either I_1 or I_2. For instance, can solve eqn **A** for I_1 : $\quad I_1 = \dfrac{V_1 - I_2 R_3}{R_1 + R_3}$ $\quad$ (eqn **C**)

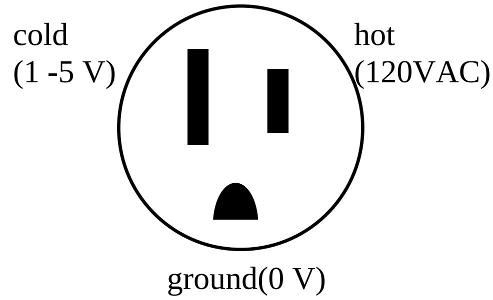
Then plug this into eqn **B**: $\quad V_2 = \left(\dfrac{V_1 - I_2 R_3}{R_1 + R_3} \right) R_3 + I_2 (R_2 + R_3)$

Can solve this for I_2 (messy!). Then plug into (**C**) to get I_1. Then plug back into (**1**) to get I_3.

Household Wiring

Wall socket = 3-prong plug

The short slot is the dangerous high-voltage one; short slot is harder to stick your finger in.



Standard electrical wiring colors:

- black = hot (120 V) "charred black"
- white = cold (few V) "white ice cold"
- green = ground (0 V) "green grass"

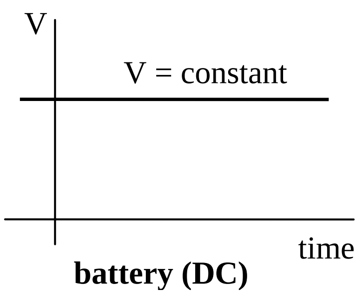
Never assume the wiring colors are correct! Always check with a voltmeter.

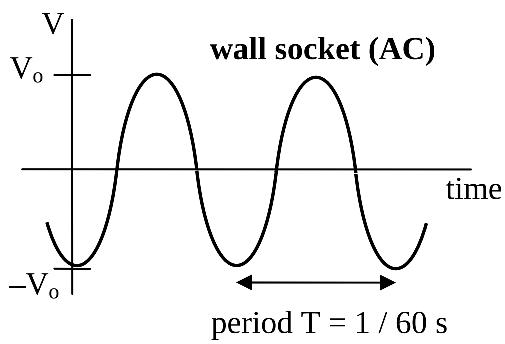
AC Voltage and Current

Batteries produce voltage that is constant in time, DC voltage.

The wall socket produces sinusoidally-varying voltage, AC voltage.

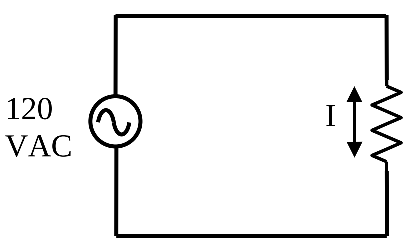
(DC originally stood for "direct current" but now it just means "constant in time". AC is short for "alternating current" but now means "sinusoidally-varying".)

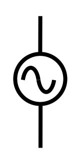




Wall socket voltage: $V = V(t) = V_o \sin\left(2\pi \frac{t}{T}\right) = V_o \sin(2\pi f t) = V_o \sin(\omega t)$

In the US, the frequency of "line voltage" is f = 60 Hz = 60 cycles per second (Recall f = 1 / T, period T = 1/60 s)





symbol for
AC voltage

AC voltage causes AC current in resistor. Current actually flows back and forth, 60 times a second.

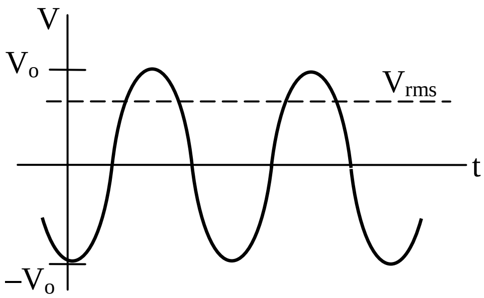
$$I = \frac{V}{R} = \frac{V_0}{R}\sin(\omega t) = I_o \sin(\omega t)$$

The instantaneous voltage is (+) as often as (–), so $\overline{V} = V_{avg} = 0$, but $|V|_{avg} \neq 0$.

Electrical engineers always report AC voltage using a kind of average called "root-mean-square" or rms average.

$$\text{VAC} = \text{"volts AC"} = V_{rms} = \sqrt{\overline{V^2}} = 120\,\text{V (in US)}$$

The average voltage V_{rms} is less than the peak voltage V_o by a factor of $\sqrt{2}$: $V_{rms} = \dfrac{V_o}{\sqrt{2}}$



Why √2 ? $V \propto \sin(\omega t)$, $V^2 \propto \sin^2(\omega t)$

Sin varies from +1 to –1 (sin = +1 → 0 → –1 → 0 → +1→ 0 → –1 → 0 → ..)

Sin^2 varies from 0 to +1 ($\sin^2$ = +1 → 0 → +1 → 0 → +1→ 0 → +1 → 0 → ..)

The average of $\sin^2$ is ½ (since average of 0 and 1 is ½) .

$$V_{rms} = \sqrt{\overline{V^2}} = \sqrt{\overline{V_o^{\,2}\sin^2(\omega t)}} = V_o\sqrt{\overline{\sin^2(\omega t)}} = V_o\sqrt{\frac{1}{2}} = \frac{V_o}{\sqrt{2}}$$

Average vs. instantaneous quantities:

Power $P = IV \overset{(AC)}{=} I_o \sin(\omega t) V_o \sin(\omega t) = I_o V_o \sin^2(\omega t)$

Since $\sin^2$ alternates between 0 and +1, the power P alteranates between 0 and $P_{max} = I_o V_o$.

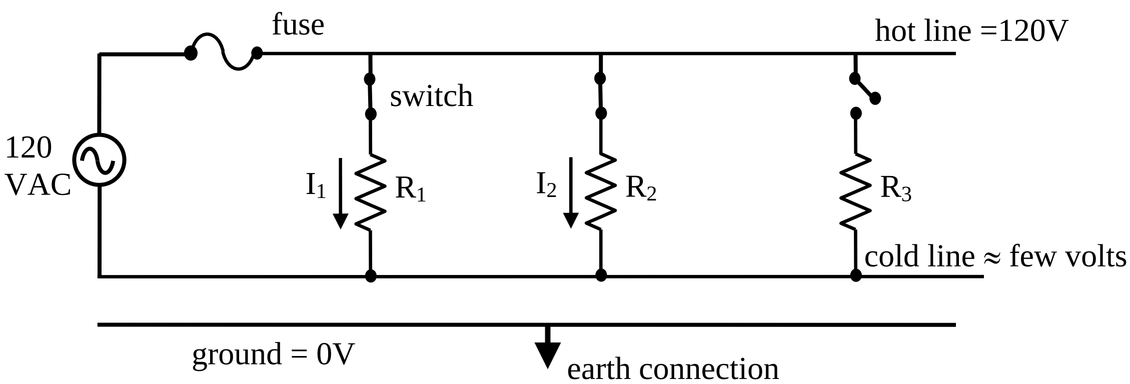
The average value of $\sin^2 = ½ \Rightarrow P_{avg} = I_o V_o \cdot \dfrac{1}{2} = \dfrac{I_o}{\sqrt{2}} \cdot \dfrac{V_o}{\sqrt{2}} = I_{rms} V_{rms}$

The old formula P = IV works OK with AC quantities if we use P_{avg} , I_{rms} , and V_{rms}.

All the old DC formulas, V = IR , $P = IV = V^2 / R = I^2 R$, still work fine for AC if we use I_{rms}, V_{rms}, and $P_{avg.}$

House Circuits

The resistance of copper wires in the walls of your home is less than 0.1 Ω. So $R_{wire} << R_{bulb} \approx 100\ \Omega$. R_{wire} is small, but not zero ⇒ wires get hot if too much current ⇒ fire hazard. So all circuits in your house have fuses or circuit breakers which automatically break the circuit if the current exceeds 15 A.



Example of voltage drop along a wire: What is the resistance of copper wire, length L = 10 m, diameter = 1 mm, $\rho = 1.7 \times 10^{-8}\ \Omega\cdot$m (typical of wires in the walls of your house.)

$$R_{wire} = \frac{\rho L}{A} = \frac{\rho L}{\pi r^2} = \frac{(1.7\times10^{-8})(10)}{\pi(0.001)^2} = 0.054\,\Omega$$

If the current through this wire is I = 15 A (close to tripping the breaker), what is the voltage drop along this wire? $V_{wire} = I\,R_{wire} = (15\text{ A})\,(0.054\ \Omega) = 0.81\text{ V}$

Cost of electricity

Power company charges for total energy used. energy = power × time (P = W / t, W = P t)

Unit of energy = kilowatt-hour (kW·h) = two hairdryers on for 1 hour.

1 kW·h costs about 10 cents (varies).

Example of energy cost. What's the bill for a 500 W hairdryer left on for 1 year?

$$1\,\text{year} \cdot 365\frac{\text{d}}{\text{y}} \cdot 24\frac{\text{h}}{\text{d}} = 8760\,\text{hours}\ , \quad \frac{\$0.10}{\text{kW}\cdot\text{h}} \cdot 8760\,\text{h} \cdot 0.5\,\text{kW} = \$438 \quad \text{(yikes!)}$$

