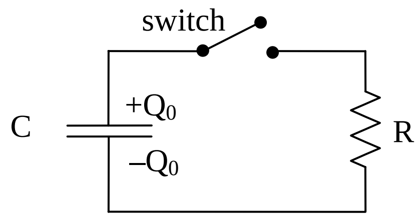
RC Circuits

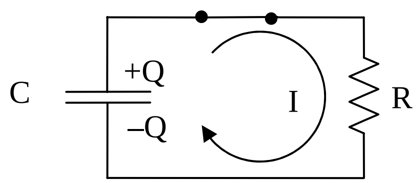
An RC circuit is a circuit with a resistor (R) and a capacitor (C). RC circuits are used to construct timers and filters.

Example 1. Very simple RC circuit: a capacitor C, charged to an initial voltage $V_0 = Q_0/C$, attached to a resistor R with a switch.



$$C = \frac{Q}{V} \Rightarrow V_0 = \frac{Q_0}{C}$$

Close the switch at time t = 0 , so current I starts to flow. The charged capacitor is acting like a battery: it produces a voltage difference across the resistor which drives the current through the resistor:



At t = 0+, $I_0 = \frac{V_0}{R}$.

$I = -\frac{dQ}{dt}$ (– sign because Q is decreasing)

$$V_{\text{across C}} = V_{\text{across R}}, \; V_C = V_R, \; \frac{Q}{C} = IR, \; \frac{Q}{C} = -\frac{dQ}{dt}\cdot R, \; \boxed{\frac{dQ}{dt} = -\frac{1}{RC}Q} \; ☆$$

RC = "time constant" = τ , has units of time

☆ is a differential equation of the form $\frac{dx}{dt} = a \cdot x$, where *a* is a constant.

This equation says: (rate of charge of x) $\propto$ x $\Leftrightarrow$ exponential solution : $x = x_0 \exp(a\,t)$

Check: $\frac{dx}{dt} = \frac{d\left(x_0 \cdot e^{at}\right)}{dt} = a \cdot x_0 \cdot e^{at} = a \cdot x$. It works!

$a > 0 \Rightarrow$ exponential growth , $a < 0 \Rightarrow$ exponential decay

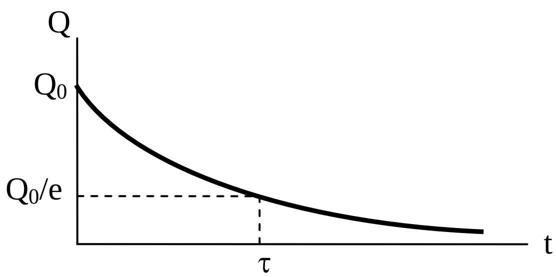
The solution to $\frac{dQ}{dt} = -\frac{1}{RC}Q$ is $Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right) = C V_0 \exp\left(-\frac{t}{\tau}\right)$

Notice that at t = 0, the formula gives $Q = Q_0$.

In time τ , Q falls by a factor of exp(–1) = 1/e $\cong$ 0.37..

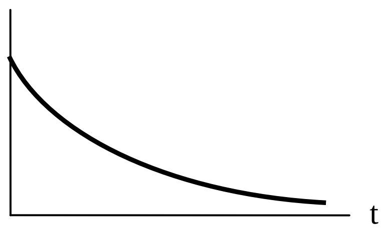
In time 2τ, Q falls by a factor of exp(–2) = (1/e)(1/e) $\cong$ 0.14..

$\Rightarrow$ Q approaches zero asymptotically, and so does V and I

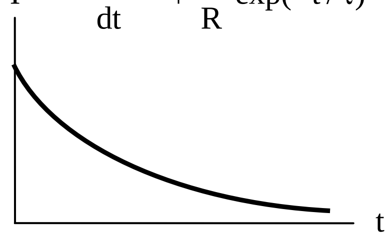


$$Q(t) = Q_0 \exp\left(-\frac{t}{RC}\right) = Q_0 \exp\left(-\frac{t}{\tau}\right)$$

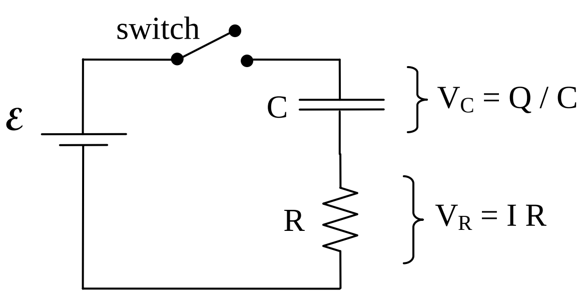
V = Q / C



$$I = -\frac{dQ}{dt} = +\frac{V_0}{R}\exp(-t/\tau)$$



Example 2: More complex RC circuit: Charging a capacitor with a battery.



Let's use symbol $\mathcal{E}$ for battery voltage ($\mathcal{E}$ short for emf) because there are so many other V's in this example.

Before switch is closed, I = 0, Q = 0.

Close switch at t = 0.

Always true that $\mathcal{E} = V_C + V_R$, by Kirchhoff's Voltage Law (Loop Law)

The charge Q on the capacitor and the voltage $V_C = Q / C$ across the capacitor cannot change instantly, since it takes time for Q to build up, so ..

At t = 0+ , Q = 0 , $V_C = 0$, $\mathcal{E} = V_C + V_R = V_R = I R \Rightarrow I_0 = \mathcal{E} / R$

Although Q on the capacitor cannot change instantly, the current I = dQ/dt can change instantly.

"Current through a capacitor" means dQ/dt . Even though there is no charge ever passing between the plates of the capacitor, there is a current going into one

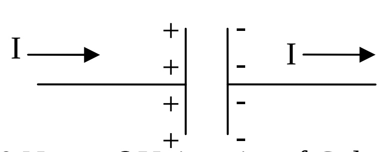
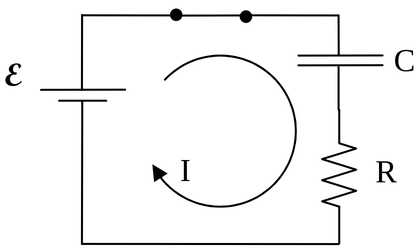


plate and the same current is coming out of the other plate, so it is as if there is a current passing through the capacitor.

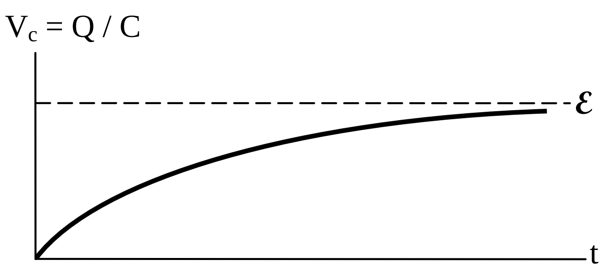


$$\mathcal{E} = V_R + V_C$$

$$\mathcal{E} = IR + \frac{Q}{C} \quad , \quad I = +\frac{dQ}{dt}$$

$$\mathcal{E} = \frac{dQ}{dt} R + \frac{Q}{C}$$

Qualitatively, as t ↑ , Q ↑ , $V_C = Q/C$ ↑ , V_R ↓ , $I = V_R / R$ ↓. After a long time , $t >> \tau = RC$, the current decreases to zero: $I = 0$, $V_C = \mathcal{E}$, $Q = C\,\mathcal{E}$



Analytic solution:

$$V_C(t) = \mathcal{E}[1 - \exp(-t/RC)]$$

$$Q(t) = \mathcal{E} \cdot C\,[1 - \exp(-t/RC)]$$

Things to remember:

- Uncharged capacitor acts like a "short" (a wire) since $V_C = Q / C = 0$.
- After a long time, when the capacitor is fully charged, it acts like an "open-circuit" (a break the wire). We must have $I_C = 0$ eventually, otherwise $Q \to \infty$, $V_C \to \infty$.

