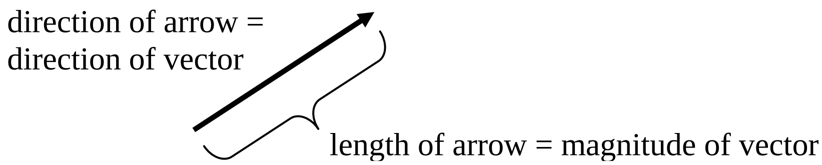
Vectors

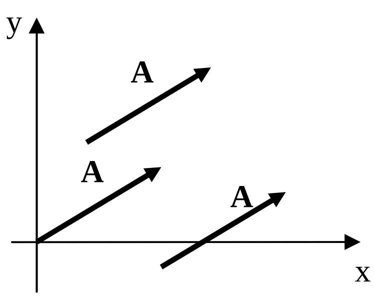
A vector is a mathematical object consisting of a magnitude (size) and a direction.

A vector can be represented graphically by an arrow:



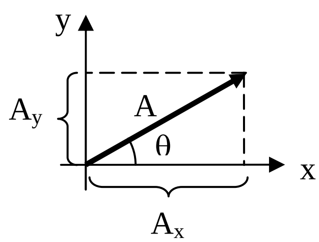
A vector quantity is written in bold (**A**) or with a little arrow overhead ($\vec{A}$)
A (no arrow, not bold) = $\left|\vec{A}\right|$ = magnitude of the vector = positive number (magnitudes are positive by definition)

Examples of vector quantities: position, velocity, acceleration, force, electric field.



If two vectors have the same direction and the same magnitude, then they are the same vector

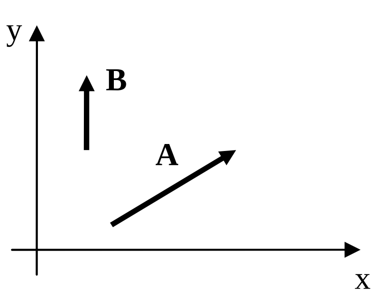
Vector = magnitude + direction (not location)

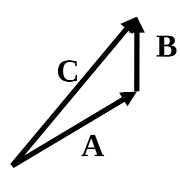


In 2D, we need 2 numbers to specify a vector $\vec{A}$:

- magnitude A and angle θ
  or
- components A_x and A_y (more on components later)

Addition of Vectors

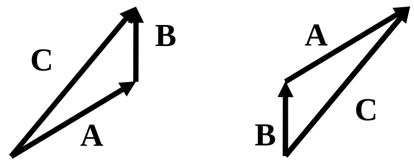




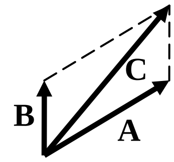
$$\vec{A}+\vec{B} = \vec{C}$$

Vector addition is commutative: $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

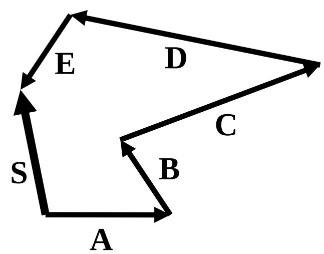
Graphical addition: "tip-to-tail" or "tail-to-head" method:



Addition by "parallelogram method" (same result as tip-to-tail method)

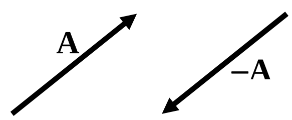


Can add lots of vectors (like steps in a treasure map: "take 20 steps east, then 15 steps northwest, then…")



$\vec{A}+\vec{B}+\vec{C}+\vec{D}+\vec{E} = \vec{S}$

Definition of negative of vector (same size, opposite direction):



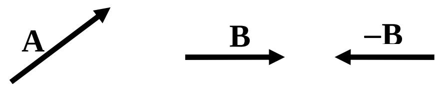
Definition of multiplication of a vector by a number:

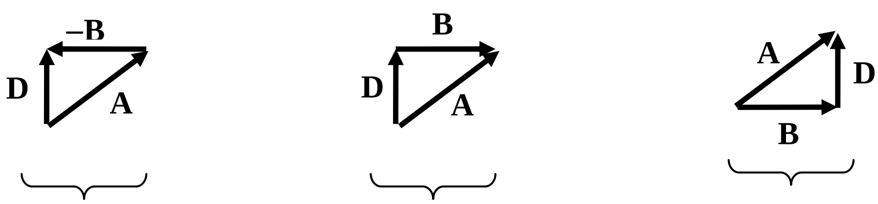
What about multiplication of a vector by a vector?
There are two different ways to define multiplication of two vectors:
(1) Dot product or scalar product $\vec{A}\cdot\vec{B}$ and (2) Cross product $\vec{A}\times\vec{B}$
These will be defined later.

Vector subtraction:

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ ← "substract" means "add negative of"

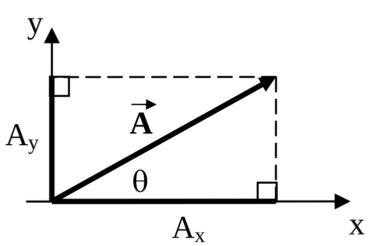
Graphically: $\vec{D} = \vec{A} - \vec{B}$





$\vec{D} = \vec{A} - \vec{B}$ is the same as $\vec{D} + \vec{B} = \vec{A}$ is same as $\vec{B} + \vec{D} = \vec{A}$

Components of a Vector



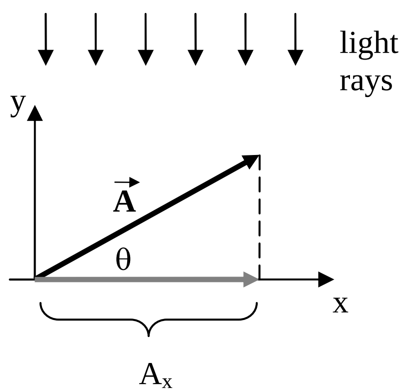
$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y}$

($\hat{x}$ = "x-hat" is the *unit vector*, explained below)

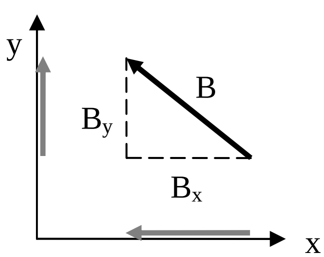
A_x = A cos θ = x-component = "projection of **A** onto x-axis"

A_y = A sin θ = y-component = "projection of **A** onto y-axis"

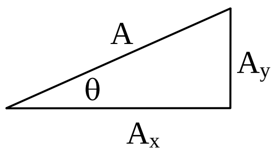
Think of the A_x as the "shadow" or "projection" of the vector **A** cast onto the x-axis by a distant light source directly "overhead" in the direction of +y.



Components are numbers, not vectors. They do not have a direction, but they do have a *sign*, a (+) or (–) sign. If the "shadow" onto the x-axis points in the +x direction, then A_x is positive.



Here, B_x is negative, because the x-projection is along the –x direction.
B_y is positive, because the y-projection is along the +y direction.



$$\cos\theta = \frac{A_x}{A} \Rightarrow A_x = A\cos\theta$$

$$\sin\theta = \frac{A_y}{A} \Rightarrow A_y = A\ \sin\theta$$

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} \qquad \tan\theta = \frac{A_y}{A_x}$$

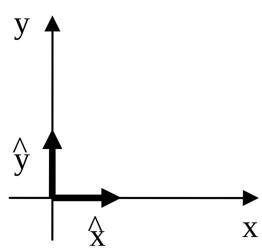
Magnitude $A = |\vec{A}|$ is positive always, but A_x and A_y can be + or – .

Unit Vectors A unit vector is a vector with length one. Unit vectors are denoted with an "hat" or carrot symbol overhead (^). The unit vector $\hat{x}$ (pronounced "x-hat") is the unit vector pointing in the +x direction. Similarly for $\hat{y}$. Do you see that $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y}$?

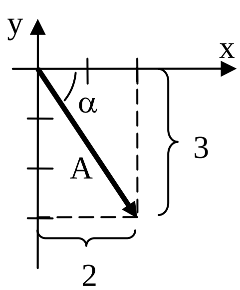
Note: A_x is a number, not a vector.

$A_x\,\hat{x}$ is a vector, because it is a number (A_x) times a vector ($\hat{x}$).

$A_x\,\hat{x} + A_y\,\hat{y}$ is the sum of two vectors



Example of vector math: $A_x = +2$, $A_y = -3$ What is the magnitude A, and the angle α with the x-axis ?



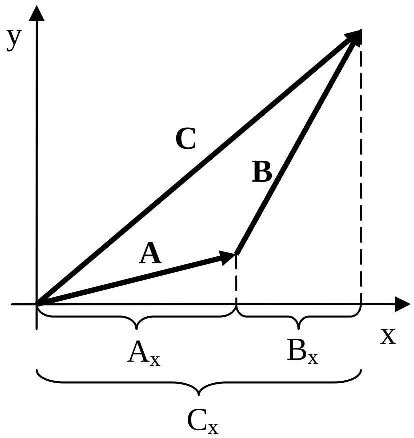
$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13} \simeq 3.6$$

$$\tan\alpha = \frac{3}{2} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^{o}$$

Vector Addition by Components:

$$\vec{C} = \vec{A} + \vec{B} \Rightarrow$$
$$C_x = A_x + B_x$$
$$C_y = A_y + B_y$$

Proof by diagram:



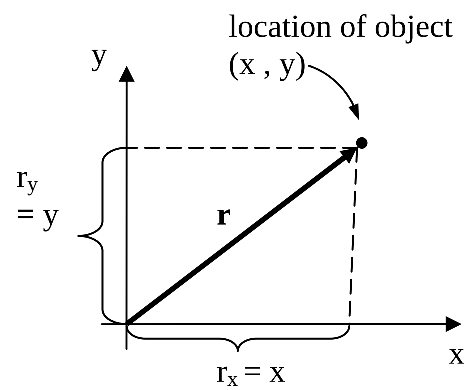
Similarly, **subtraction by components:**

$$\vec{D} = \vec{A} - \vec{B} \quad \Rightarrow \quad D_x = A_x - B_x$$
$$D_y = A_y - B_y$$

Position, Velocity, and Acceleration Vectors

Velocity is a vector quantity; it has a magnitude, called the speed, and a direction, which is the direction of motion. Position is also a vector quantity. Huh? What do we mean by the magnitude and direction of position? How can position have a direction?

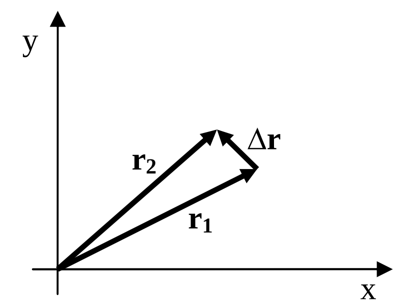
In order to specify the position of something, we must give its location in some coordinate system, that is, its location relative to some origin. We define the position vector **r** as the vector which stretches from the origin of our coordinate system to the location of the object. The x- and y-components of the position vector are simply the x and y coordinates of the position. Notice that that the position vector depends on the coordinate system that we have chosen.

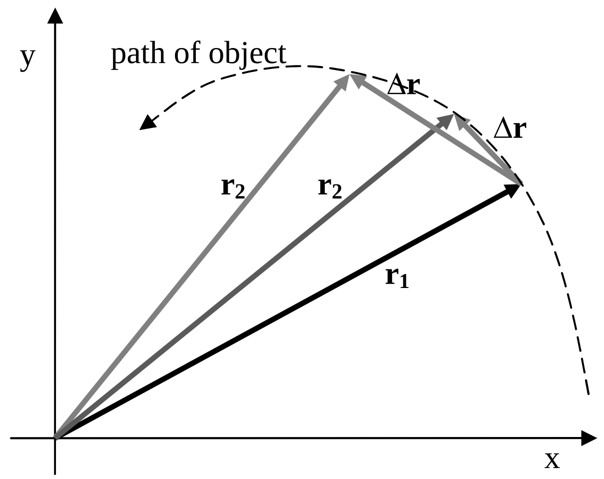


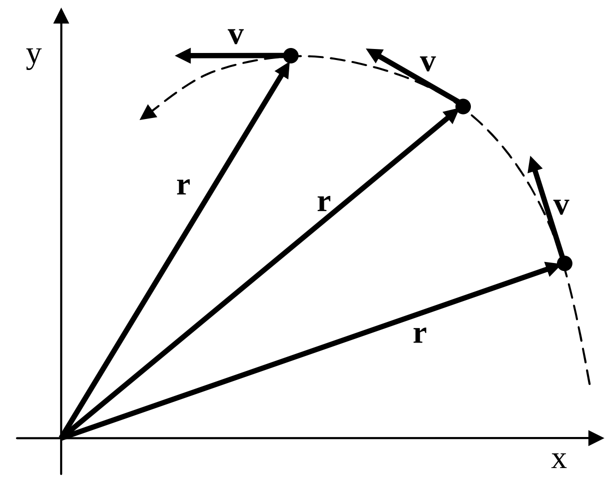
If the object is moving, the position vector is a function of time **r** = **r**(t). Consider the position vector at two different times t_1 and t_2, separated by a short time interval $\Delta t = t_2 - t_1$. (Δt is read "delta-t") The position vector is initially $\mathbf{r}_1$, and a short time later it is $\mathbf{r}_2$. The change in position during the interval Δt is the vector $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Notice that, although $\mathbf{r}_1$ and $\mathbf{r}_2$ depend on the choice of the origin, the change in position $\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is independent of choice of origin. Also, notice that change in something = final something – initial something.

In 2D or 3D, we define the velocity vector as $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$.

As Δt gets smaller and smaller, $\mathbf{r}_2$ is getting closer and closer to $\mathbf{r}_1$, and $\Delta\mathbf{r}$ is becoming tangent to the path of the object. Note that the velocity $\mathbf{v}$ is in the same direction as the infinitesimal $\Delta\mathbf{r}$, since the vector $\mathbf{v}$ is a positive number $(1/\Delta t)$ times the vector $\Delta\mathbf{r}$. Therefore, the velocity vector, like the infinitesimal $\Delta\mathbf{r}$, is always tangent to the trajectory of the object.







The vector equation $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$ has x- and y-components. The component equations are

$v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$, $v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$. Any vector equation, like $\vec{A} = \vec{B} + \vec{C}$, is short-hand notation for 2 or 3 component equations: $A_x = B_x + C_x$, $A_y = B_y + C_y$, $A_z = B_z + C_z$

The change in velocity between two times t_1 and t_2 is $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$ (remember that change is always final minus initial). We define the acceleration vector as $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t}$. As we mentioned in the chapter on 1D motion, the direction of the acceleration is the same as the direction of $\Delta\mathbf{v}$. The direction of the acceleration is NOT the direction of the velocity, it is the "direction towards which the velocity is tending", that is, the direction of $\Delta\mathbf{v}$.

We will get more experience thinking about the velocity and acceleration vectors in the next few chapters.

