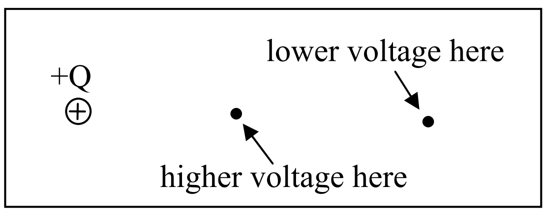
Voltage (= Electric Potential)

An electric charge alters the space around it. Throughout the space around every charge is a vector thing called the *electric field*. Also filling the space around every charge is a scalar thing, called the *voltage* or the *electric potential*. Electric fields and voltages are two different ways to describe the same thing.

(Note on terminology: The text book uses the term "electric potential", but it is easy to confuse electric potential with "potential energy", which is something different. So I will use the term "voltage" instead.)

Voltage overview

The voltage at a point in empty space is a number (not a vector) measured in units called *volts* (V) . Near a positive charge, the voltage is high. Far from a positive charge, the voltage is low. Voltage is a kind of "electrical height". Voltage is to charge like height is to mass. It takes a lot of energy to place a mass at a great height. Likewise, it takes a lot of energy to place a positive charge at a place where there voltage is high.

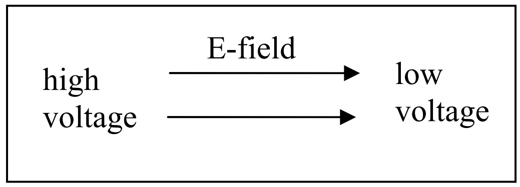


Only changes in voltage ΔV between two different locations have physical significance. The zero of voltage is arbitrary, in the same way that the zero of height is arbitrary.
We define ΔV in 2 equivalent ways:

- $\Delta V = \dfrac{\Delta U_{\text{of q}}}{q}$ = change in potential energy of a test charge divided by the test charge
- $\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{r}$

For constant E-field, this integral simplifies to $\Delta V = -\vec{E} \cdot \Delta\vec{r}$ ($\Delta\mathbf{r}$ = change in position)

The electric field is related to the voltage in this way: Electric field is the rate of change of voltage with position. E-field is measured in units of N/C, which turn out to be the same as volts per meter (V/m). E-fields points from high voltage to low voltage. Where there is a big E-field, the voltage is varying rapidly with distance.



In order to understand these strange, abstract definitions of voltage, we must review work and potential energy

Work and Potential Energy (U)

Definition of work done by a force: consider an object pulled or pushed by a constant force $\vec{F}$.
While the force is applied, the object moves through a displacement of $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$.

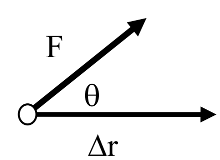
Notice that the direction of displacement is not the same as the direction of the force, in general.

Work done by a force F = $\boxed{W_F \equiv \vec{F} \cdot \Delta\vec{r} = F\,\Delta r \cos\theta = F_{||}\,\Delta r}$ (constant **F**)

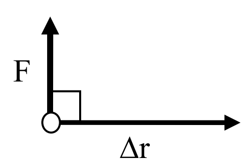
$F_{||}$ = component of force along the direction of displacement

If the force F varies during the displacement (or the displacement is not a straight line), then we must use the more general definition of work done by a force $\boxed{W_F \equiv \int \vec{F} \cdot d\vec{r}}$

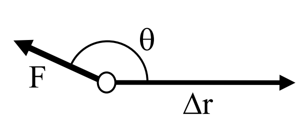
Work is not a vector, but it does have a sign (+) or (-). Work is positive, negative, or zero, depending on the angle between the force and the displacement.



$\theta < 90$, W positive



$\theta = 90$, W = 0

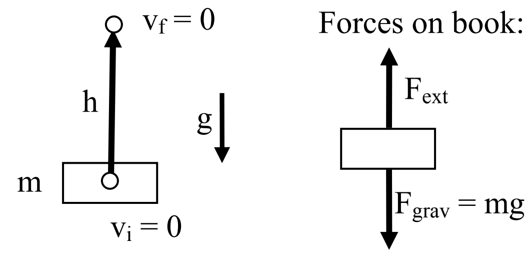


$\theta > 90$, W negative

Definition of Potential Energy U: Associated with *conservative forces*, such as gravity and electrostatic force, there is a kind of energy of position called potential energy. The change in potential energy ΔU of a system is defined to be the negative of the work done by the "field force", which is the work done by an "external agent" opposing the field.

$$\Delta U \equiv W_{ext} = -W_{field}$$

This is best understood with an example: A book of mass m is lifted upward a height h by an "external agent" (a hand which exerts a force to oppose the force of gravity). The force of gravity is the "field". In this case, the work done by the hand is $W_{ext} = +mgh$. The work done by the field (gravity) is $W_{field} = -mgh$. The change in the potential energy of the earth/book system is $\Delta U = W_{ext} = -W_{field} = +mgh$. The work done by the external agent went into the increased gravitational potential energy of the book. (The initial and final velocities are zero, so there was no increase in kinetic energy.)



A conservative force is force for which the amount of work done depends only on the initial and final positions, not on the path taken in between. Only in the case that the work done by the field in independent of the path, does it make any sense to associate a change in energy with a change in position.

Potential energy is a useful concept because (if there is no friction, no dissipation)

$$\Delta K + \Delta U = 0 \iff K + U = \text{constant} \ \text{(no dissipation)}$$

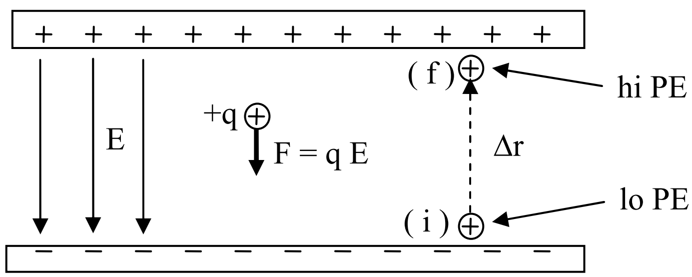
$$(K = \text{kinetic energy} = \tfrac{1}{2} m v^2)$$

Voltage

We define *electrostatic potential energy* (not to be confused with electrostatic potential or voltage) in the same way as we defined gravitational potential energy, with the relation $\Delta U = W_{ext} = -W_{field}$. Consider two parallel metal plates (a capacitor) with equal and opposite charges on the plates which create a uniform electric field between the plates. The field will push a test charge +q toward the negative plate with a constant force of magnitude F = q E. (The situation is much like a mass in a gravitational field, but there is no gravity in this example.) Now imagine grabbing the charge with tweezers (an external agent) and pulling the charge +q a displacement $\Delta\mathbf{r}$ against the electric field toward the positive plate. By definition, the change in electrostatic potential energy of the charge is

$$\Delta U = U_f - U_i = +W_{ext} = -W_{field} = -\vec{F}_{field}\cdot\Delta\vec{r} = -q\,\vec{E}\cdot\Delta\vec{r}$$

I recommend that you do not try to get the signs from the equations – it's too easy to get confused. Get the sign of ΔU by asking whether the work done by the external agent is positive or negative and apply $\Delta U = +W_{ext}$.



If the E-field is not constant, then the work done involves an integral

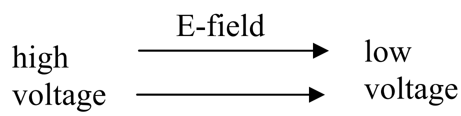
$$\Delta U = U_f - U_i = -W_{field} = -\int_i^f \vec{F}_{field}\cdot d\vec{r} = -q\int_i^f \vec{E}\cdot d\vec{r}\,.$$

Now we are ready for the definition of voltage difference between two points in space. Notice that the change in PE of the test charge q is proportional to q, so the ratio ΔU/q is independent of q. Recall that electric field is defined as the force *per* charge : $\vec{E} \equiv \dfrac{\vec{F}_{on\,q}}{q}$. Similarly, we define the voltage difference ΔV as the change in PE *per* charge:

$$\boxed{\Delta V \equiv \frac{\Delta U}{q} = -\int_i^f \vec{E}\cdot d\vec{r}}\quad , \quad \text{or} \quad \Delta U = q\,\Delta V$$

Remember that the E-field always points from high voltage to low voltage:

$$\Delta V = -\vec{E}\cdot\Delta\vec{r} \qquad (\text{if } \mathbf{E} = \text{constant})$$



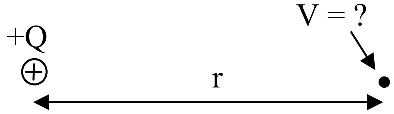
If $\vec{E} \parallel \Delta\vec{r}$, then $\Delta V = -\vec{E}\cdot\Delta\vec{r} = (-) \quad \Rightarrow \quad V_f - V_i < 0 \, , \, V_i > V_f$

To say that "the voltage at a point in space is V" means this: if a test charge q is placed at that point, the potential energy of the charge q (the work required to place the charge there) is $U = q\,V$. If the charge is moved from one place to another, the change in PE is $\Delta U = q\,\Delta V$.

Only *changes* in PE and *changes* in V are physically meaningful. We are free set the zero of PE and V anywhere we like.

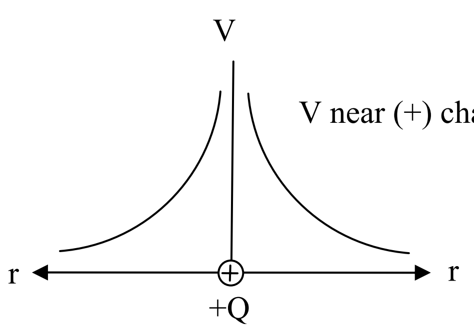
Units of voltage = $[V] = \dfrac{\text{energy}}{\text{charge}} = \dfrac{\text{joule}}{\text{coulomb}} = \text{volt (V)}$.   1 V = 1 J/C

Voltage near a point charge

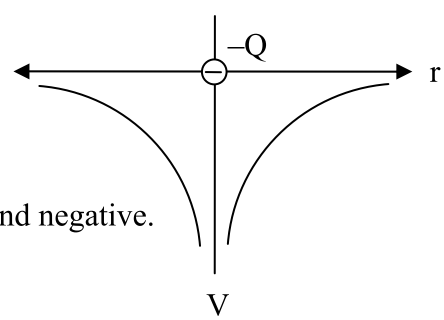


Answer: $\boxed{V(r) = \dfrac{kQ}{r}}$

Notice that this formula gives $V = 0$ at $r = \infty$. When dealing with point charges, we always set the zero of voltage at $r = \infty$.

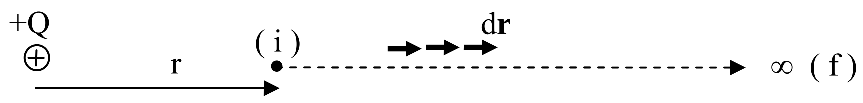


V near (+) charge is large and positive.



V near (–) charge is large and negative.

Proof:



$\vec{E} \parallel d\vec{r} \quad \Rightarrow \quad \vec{E}\cdot d\vec{r} = +E\,dr$, so we have

$$\Delta V = V_f - V_i = \underbrace{V(r=\infty)}_{0} - V(r) = -\int_r^{\infty} E\,dr \;\Rightarrow$$

$$V(r) = +\int_r^{\infty} E\,dr = \int_r^{\infty} \frac{kQ}{r^2}dr = kQ\left(-\frac{1}{r}\right)\Bigg|_r^{\infty} = \frac{kQ}{r} \qquad \text{Done.}$$

Voltage due to several charges

If we have several charges Q_1, Q_2, Q_3, … , the voltage at a point near the charges is

$$V_{tot} = V_1 + V_2 + V_3 + \ldots = \sum_i V_i = \sum_i \frac{kq_i}{r_i} \;\; \underline{\text{or}} \;\; \int k\frac{dq}{r}$$

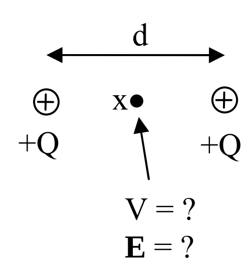
Proof: $\Delta V = -\int \vec{E}_{tot}\cdot d\vec{r} = -\int (\vec{E}_1 + \vec{E}_2 + \ldots)\cdot d\vec{r} = \Delta V_1 + \Delta V_2 + \ldots$

Voltages add like numbers, not like vectors.

What good is voltage?

- <u>Much</u> easier to work with V's (scalars) than with $\vec{E}$'s (vectors).
- Easy way to compute PE.

Voltage example: Two identical positive charges are some distance d apart. What is the voltage at point x midway between the charges? What is the E-field midway between the charges? How much work is required to place a charge +q at x?



$V_{tot} = V_1 + V_2 \;=$

$$V_{tot} = V_1 + V_2 = \frac{kQ}{(d/2)} + \frac{kQ}{(d/2)} = \frac{2kQ}{d} + \frac{2kQ}{d} = \frac{4kQ}{d}$$

The E-field is zero between the charges (Since $\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 = 0$. Draw a picture to see this!)

The work required to bring a test charge +q from far away to the point x is positive, since it is hard to put a (+) charge near two other (+) charges. You have to push to get the +q in place. The work done is

$$W_{ext} = \Delta U = +q\,\Delta V, \quad \text{where } \Delta V = V_{final} - V_{initial} = V(\text{at } x) - \underbrace{V(\text{at } \infty)}_{0} = \frac{4kQ}{d}$$

$$W_{ext} = \frac{4kqQ}{d}$$

Units of electron-volts (eV)

The SI units of energy is the joule (J). 1 joule = 1 newton·meter = 1N·m Another, non-SI unit of energy is the electron-volt (eV), often used by chemists. The eV is a very convenient unit of energy to use when working with the energies of electrons or protons.

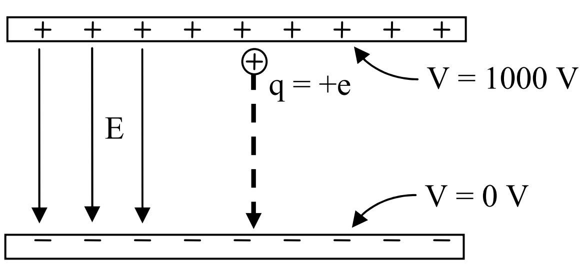
From the relation $\Delta U = q\,\Delta V$, we see that energy has the units of charge × voltage. If the charge q = 1 e = |charge of the electron| and ΔV = 1 volt, then $\Delta U = q\,\Delta V$ = 1 e × 1V = a unit of energy called an "eV". Notice that the name "eV" reminds you what the unit is: it's an "e" times a "V" = 1 e × 1 volt.

How many joules in an eV? 1 eV = 1 e × 1V = $(1.6 \times 10^{-19}$ C)(1 V) = 1.6×10^{-19} J

$$1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$$

If q = e (or a multiple of e), it is easier to use units of eV instead of joules when computing (work done) = (change in PE).

Example of use of eV. A proton, starting at rest, "falls" from the positive plate to the negative plate on a capacitor. The voltage difference between the plates is ΔV = 1000 V. What is the final KE of the proton (just before it hits the negative plate)?

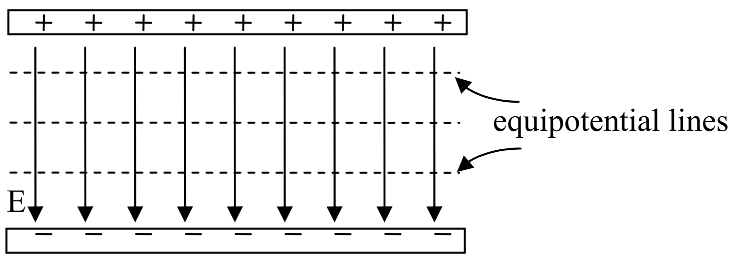


As the proton falls, it loses PE and gains KE.

$$|\Delta K| = |\Delta U| = |q\,\Delta V| = 1e \times 1000 \text{ V} = 1000 \text{ eV}$$

"Equipotential Lines" = constant voltage lines

Given **E**, we can compute $\Delta V = -\int \vec{E}\cdot d\vec{r} = -\vec{E}\cdot\Delta\vec{r}$ (if **E** constant) . Notice that if the displacement Δ**r** is perpendicular to the direction of **E**, then $\Delta V = -\vec{E}\cdot\Delta\vec{r} = 0$. A surface of constant V is one along which $\Delta V = 0$. Equipotential (constant voltage) lines are always at right angles to the electric field.



Computing E from V

Given **E**, we can compute $\Delta V = -\int \vec{E}\cdot d\vec{r} = -\vec{E}\cdot\Delta\vec{r}$ (if **E** constant) .

Given V = V(x, y, z), how do we get **E** ?

Suppose $\Delta\vec{r} \parallel \vec{E}$ and Δr very small so that $E \cong$ constant, then

$$\Delta V = -\vec{E}\cdot\Delta\vec{r} = -E\,\Delta r \quad\Rightarrow\quad E = -\frac{\Delta V}{\Delta r} = -\frac{dV}{dr}$$

"E is the rate of change of V": but $E = -\dfrac{dV}{dr}$ only if the r-axis is the direction of **E**.

Suppose $\Delta\vec{r} = \Delta x\,\hat{x}$ is along the x-axis, but not necessarily along **E**.

$$\Rightarrow \vec{E}\cdot\Delta\vec{r} = \vec{E}\cdot\hat{x}\,\Delta x = E_x\,\Delta x = -\Delta V \quad\Rightarrow\quad E_x = -\frac{\Delta V}{\Delta x} = -\frac{dV}{dx}\ , \quad E_y = -\frac{dV}{dy}\ \text{, etc}$$

Actually, must take "partial derivative" $E_x = -\dfrac{\partial V}{\partial x}$ ← means hold y, z constant, take derivative w.r.t x

Gradient operator: $\vec{E} = -\nabla V = -\left(\dfrac{\partial V}{\partial x}\hat{x} + \dfrac{\partial V}{\partial y}\hat{y} + \dfrac{\partial V}{\partial z}\hat{z}\right)$

A Metal object in equilibrim is an equipotential

Metal objects (conductors) in electrostatic equilibrium are always equipotentials (V = constant everywhere inside and on the surface).