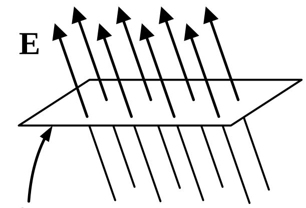
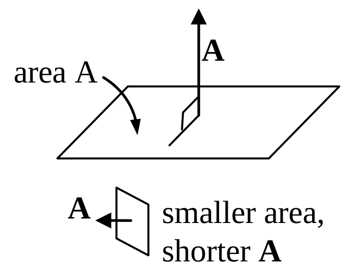
Gauss's Law



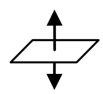
Gauss's Law is one of the 4 fundamental laws of electricity and magnetism called Maxwell's Equations. Gauss's law relates charges and electric fields in a subtle and powerful way, but before we can write down Gauss's Law, we need to introduce a new concept: the *electric flux* Φ through a surface.

Consider an imaginary surface which cuts across some E-field lines. We say that there is some electric flux through this surface. To make the notion of flux precise, we must first define a *surface vector*.

Definition: surface vector **A** = $\vec{A} = A\,\hat{n}$, associated with a flat surface of area A.
Magnitude of vector **A** = area A of surface.
Direction of vector **A** = direction perpendicular (normal) to surface = direction of unit normal $\hat{n}$.

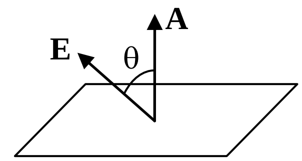


Notice that there is an ambiguity in the direction $\hat{n}$. Every flat surface has <u>two</u> perpendicular directions.



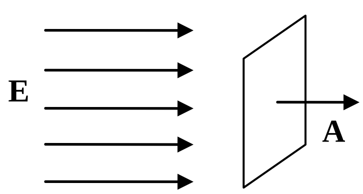
The electric flux Φ through a surface A is defined as

$$\boxed{\Phi = \vec{E}\cdot\vec{A} = E\,A\cos\theta}$$ (for **E** = constant, surface flat)

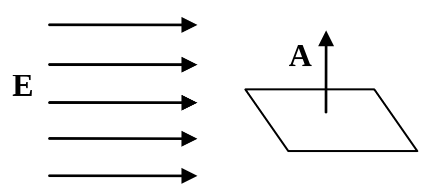


The flux Φ has the following geometrical interpretation:
| flux | $\propto$ the number of electric field lines crossing the surface.

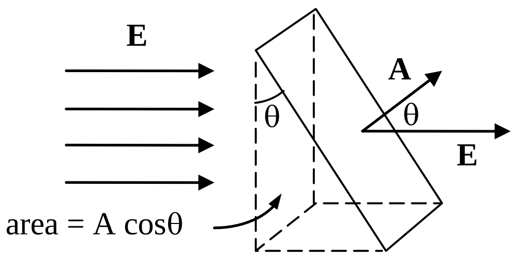
Think of the E-field lines as rain flowing threw an open window of area A. The flux is a measure of the amount of rain flowing through the window. To get a big flux, you need a large E, a large A, <u>and</u> you need the area perpendicular to the E-field vector, which means the area vector **A** is <u>parallel</u> to **E**. (In the rain analogy, you need the window to be facing the rain direction.)



$\theta = 0$, $\cos\theta = 1$, Φ max

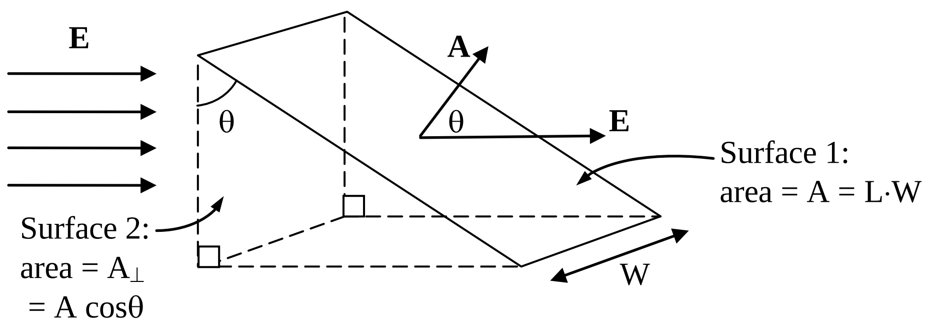


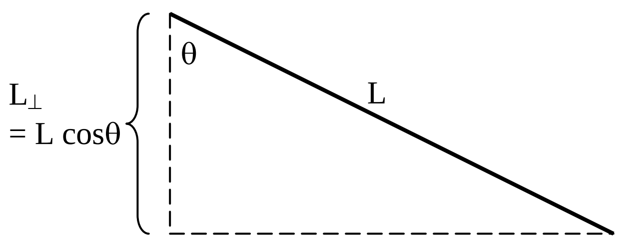
$\theta = 90^o$, $\cos\theta = 0$, $\Phi = 0$



(A cosθ) is the projection of the area A onto the plane perpendicular to **E**. The plane perpendicular to **E** is the area which "faces the rain". Only the area facing the rain contributes to the flux.

Let's consider this flux business in a little more detail. In the diagram below, we have a constant electric field **E**, passing through surface 1, represented by vector **A,** tilted at angle θ. [We use bold font **A** for vectors.] This tilted surface 1 has area $|\vec{A}| = A = L \cdot W$. The *projection* of this surface 1 onto the plane perpendicular to the E-field is surface 2, which has an area that we call $A_\perp$ (for area of surface *perpendicular* to direction of **E**). The area of this plane, this surface 2, is
$A_\perp = L_\perp \cdot W = L \cdot \cos\theta \cdot W = A \cos\theta$. So we have $A_\perp = A \cos\theta$.



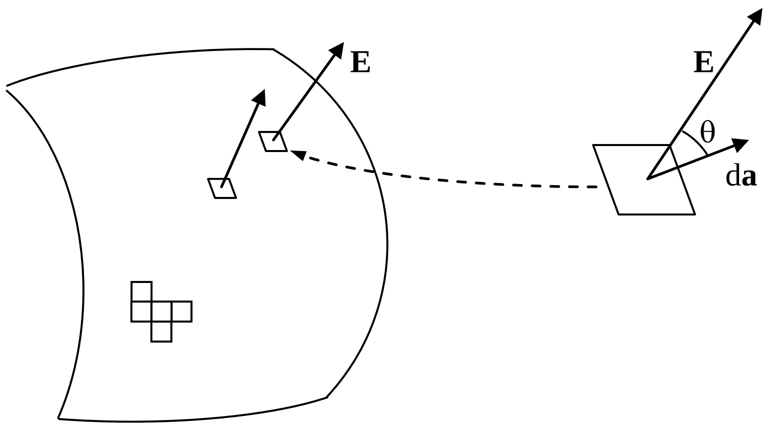


Now we are going to show that the flux $\Phi = \vec{E} \bullet \vec{A}$ through a surface **A** is proportional to the number of field lines passing through the surface. Recall that, from the definition of a field line diagram, the magnitude of the E-field is proportional to the density of the lines: $|\vec{E}| = E \propto (\#\text{ field lines})/A_\perp = N / A_\perp$ (N is the number of field lines through the area $A_\perp$). So, we have $N \propto E \cdot A_\perp$. Now, we showed above that $A_\perp = A\cos\theta$, so we have $N \propto E \cdot A_\perp = E \cdot A\cos\theta = \vec{E} \bullet \vec{A}$. Done! The number N of field lines through a surface is proportional to the flux $\Phi = \vec{E} \bullet \vec{A}$.

We can now see that, since the same number of E-field lines pass through both surfaces 1 and 2, they must have the same magnitude flux. The math shows the same thing: For surface 1, $\Phi_1 = E\,A\cos\theta$. For surface 2, $\Phi_2 = E\,A_\perp = E\,A\cos\theta$.

Now, the formula $\Phi = \vec{E}\bullet\vec{A}$ is a special case formula: it only works if the surface is flat and the E-field is constant. If the E-field varies with position and/or the surface is not flat, we need a more general definition of flux:

$$\boxed{\Phi = \int \vec{E}\cdot d\vec{a} \quad = \text{"surface integral of } \mathbf{E}\text{"}}$$



To understand a surface integral, do this: in your imagination, break the total surface up into many little segments, labeled with an index i. The surface vector of segment i is $d\vec{a}_i$. If the segment is very, very tiny, it is effectively flat and the E-field is constant over that tiny surface, so we can use our special case formula $\Phi = \vec{E}\cdot\vec{A}$.

The flux through segment i is therefore $\Phi_i = \vec{E}_i \cdot d\vec{a}_i$. (**E**$_i$ is the field at the segment i)

The total flux is the sum: $\Phi = \sum_i \vec{E}_i \cdot d\vec{a}_i \;\rightarrow\; \int \vec{E}\cdot d\vec{a}$

(In the limit that the segments become infinitesimal, there are an infinite number of segments and the sum becomes an integral.)

In general, computing surface integral $\int \vec{E}\cdot d\vec{a}$ can be extremely messy. So why do we care about this thing called the electric flux? The electric flux is related to charge by Gauss's Law.

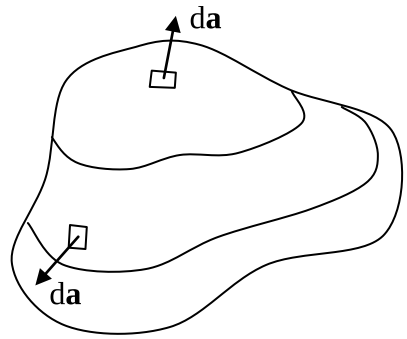
Gauss's Law (the 1st of 4 Maxwell's Equations)

$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

In words, the electric flux through any <u>closed</u> surface S is a constant ($1/\varepsilon_0$) times the total charge inside S.

$$\oint \vec{E}\cdot d\vec{a} = \textit{closed} \text{ surface integral}$$

"closed"



A surface is closed if it has no edges, like a sphere. For a closed surface, the direction of d**a** is always the <u>outward</u> normal.

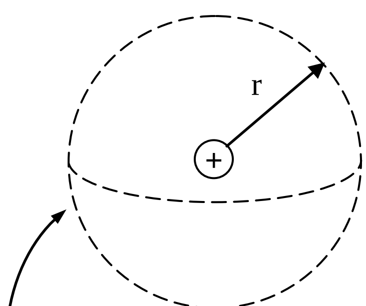
The constant ε_0 is related to k by $k = \dfrac{1}{4\pi\varepsilon_0}$.

$$F_{\text{coul}} = \frac{k\,q_1 q_2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$\varepsilon_0 = 8.85\times 10^{-12}$ (SI units)

Gauss' Law can be derived from Coulomb's Law <u>if</u> the charges are stationary, but Gauss's Law is more general than Coulomb's Law. Coulomb's Law is <u>only</u> true if the charges are stationary. Gauss's Law is <u>always</u> true, whether or not the charges are moving.

It is easy to show that Gauss's Law is consistent with Coulomb's Law. From Coulomb's Law, the E-field of a point charge is $E = \dfrac{kQ}{r^2} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}$. We get the same result by applying Gauss's Law:



imaginary spherical surface S, radius r

$$\oint_S \vec{E}\cdot d\vec{a} = \oint E\,da \quad \text{(since } \mathbf{E} \text{ is } \textit{parallel} \text{ to d}\mathbf{a} \text{ on S)}$$

$$= E\oint da \quad \text{(since E is constant on S)}$$

$$= E\,A$$

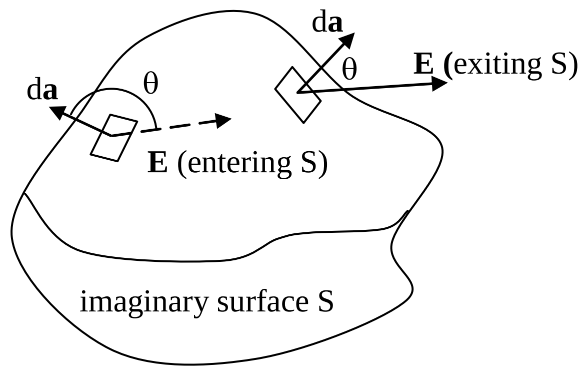
$$= E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \quad \text{(says Mr. Gauss)}$$

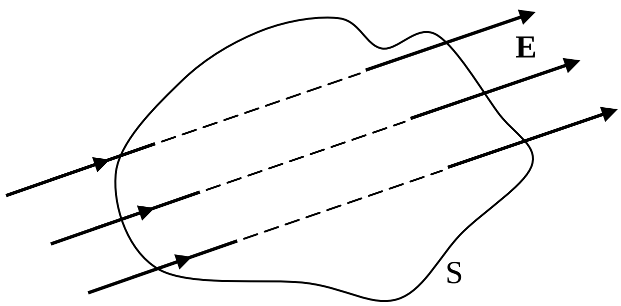
Solving for E, we have $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}$. Done.

When viewed in terms of field lines, Gauss's Law is almost obvious (after a while). Recall that flux is proportional to the number of field lines passing through the surface. Notice also that flux can be positive or negative depending on the angle θ between the E-field vector and the area vector. Where the field lines exit a closed surface, the flux there is positive; where the field lines enter a closed surface, the flux there is negative.

So the total flux through a closed surface is proportional to
[(# field lines exiting) – (# field lines entering)]

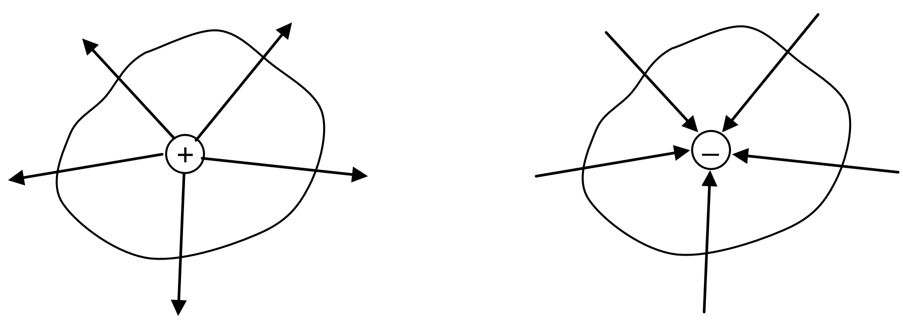
If a closed surface S encloses no charges, then the number of lines entering must equal the number of lines exiting, since there are no charges inside for the field lines to stop or start on.





← $\oint_S \vec{E}\cdot d\vec{a} = 0$

So only charges *inside* the surface can contribute to the flux through the surface. Positive charges inside produce positive flux; negative charges produce negative flux.



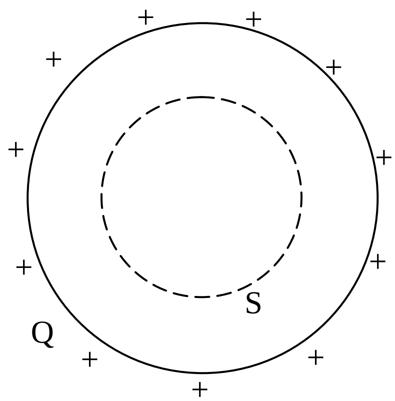
The net flux is due only to the net charge inside: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_0}$.

Using Gauss's Law to solve for the E-field

Gauss's Law is *always true* (it's a LAW). But it is not always useful. Only in situations with very high symmetry is it easy to compute the flux integral $\oint \vec{E}\cdot d\vec{a}$. In these few cases of high symmetry, we can use Gauss's Law to compute the E-field.

Example of Spherical Symmetry:

Compute E-field everywhere inside a uniformly-charged spherical shell.



By symmetry, **E** must be radial (along a radius), so E = E(r). We choose an imaginary surface S concentric with and inside the charged sphere.

Since the E-field is radial and the surface vector d**a** on S is also radial, we have $\vec{E}\cdot d\vec{a} = E\,da$. (The dot product of the parallel vectors is just the product of the magnitudes.) So we have

$$\oint_S \vec{E}\cdot d\vec{a} \;=\; \oint E\,da \;=\; E\oint da = E\,A\,,$$

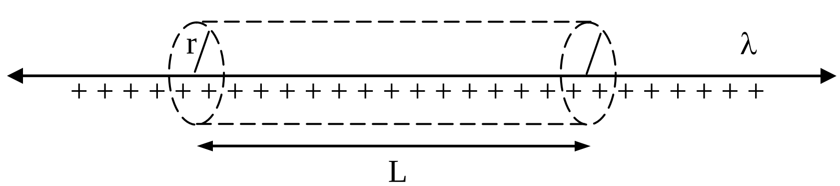
where A is the area of surface S. We are able to take E outside the integral only because E = E(r) and so E = constant on the surface S.

Gauss's Law says $\oint_S \vec{E}\cdot d\vec{a} \;=\; \dfrac{q_{enc}}{\varepsilon_0} \;=\; 0$, so we have E A = 0 , E = 0.

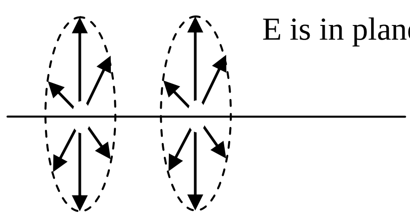
Conclusion: **E** = 0 everywhere inside a hollow uniform sphere of charge.

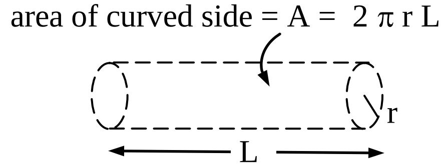
If you draw the spherical gaussian surface S outside the charged shell, you can quickly show that $\vec{E} \;=\; \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\,\hat{r}$ everywhere outside the shell. The E-field outside a uniform shell of charge, or outside any spherically symmetric charge distribution, is exactly the same as if all the charge was concentrated at the center.

Example of Cylindrical Symmetry:
Compute E-field around an infinite line of charge with charge per length = λ.



By symmetry, **E** is in the cylindrically radial direction and E = E(r).





$$\oint_S \vec{E}\cdot d\vec{a} = \underbrace{\int_{\text{ends}} \vec{E}\cdot d\vec{a}}_{0} + \int_{\text{side}} \vec{E}\cdot d\vec{a} = E\int_{\text{side}} da = EA_{\text{side}}$$

$$= E(2\pi r L)$$

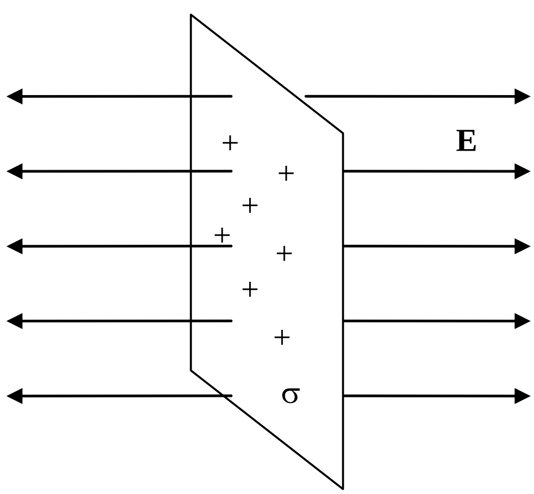
The charge inside the gaussian surface is (charge/length)×length = λ L, so Gauss gives gives

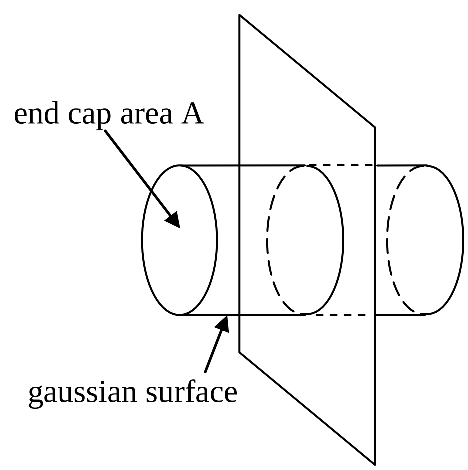
$$\oint_S \vec{E}\cdot d\vec{a} = E(2\pi r \cancel{L}) = \frac{q_{enc}}{\varepsilon_0} = \frac{\lambda \cancel{L}}{\varepsilon_0} \quad \Rightarrow \quad \boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

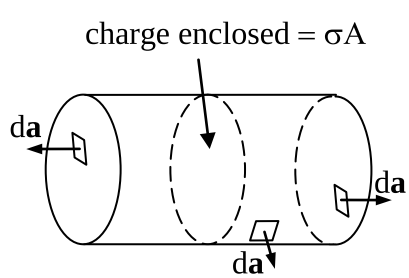
Example of Planar Symmetry:

Compute the E-field near an infinite plane of charge with $\sigma = \frac{Q}{A} = \text{charge per area}$.

By symmetry, the E-field must be perpendicular to the plane (either away or towards).





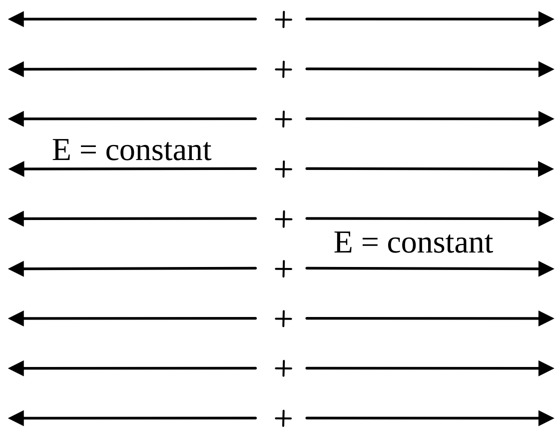


On end caps, $\vec{E}\cdot d\vec{a} = E\,da$

On curved side, $\vec{E}\cdot d\vec{a} = 0$

$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} \quad \Rightarrow \quad 2E\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

$E = \frac{\sigma}{2\varepsilon_0}$ = constant, regardless of position!



uniform plane of charge
(seen edge-on)

Conductors in Electrostatic Equilibrium

"Electrostatic equilibrium" means that all charges are stationary; so the net force on every charge must be zero (otherwise the charge would be accelerating).

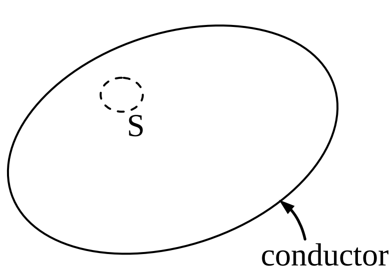
Useful facts about metals (conductors) *in electrostatic equilibrium*:

- The electric field in the *interior* of a metal must be zero.

The E-field must be zero in the interior, otherwise the conduction electrons in the metal would feel a force **F** = q **E** = –e **E** and would move in response. Electrons in motion would mean we are not in electrostatic equilibrium. In a conductor, the charges arrange themselves so the **E** = 0 everywhere in the interior (otherwise, the charges are not yet in equilibrium and continue to move).

- The interior of a conductor in equilibrium can have no *net* charge (electrons and protons must have equal density).

Proof from Gauss's Law: consider <u>any</u> closed surface within the conductor,



$$\oint_S \vec{E} \cdot d\vec{a} = 0 \quad (\text{since } E = 0)$$

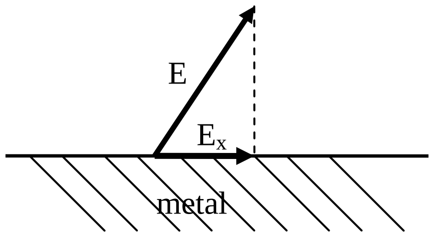
$$\Rightarrow \quad q_{enclosed} = 0$$

- Any net charge on the conductor resides only on the *surface* of the conductor.

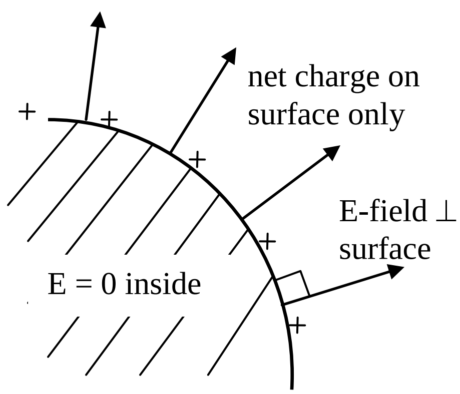
We just showed that no net charge can exist in the interior, so it must be on the surface.

• The electric field must be *perpendicular* to the surface of the conductor.

The E-field must be perpendicular to the surface (in electrostatic equilibrium), otherwise the *component* of the E-field *along* the surface would push electrons along the surface causing movement of charges (and we would not be in equilibrium).



On the surface of metal, if E_x was not zero, there would be a force $F_x = q\,E_x = -e\,E_x$ on electrons in the metal pushing them along the surface.

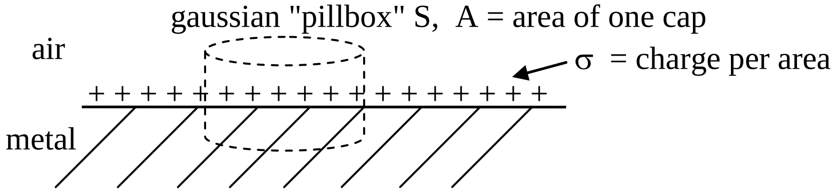


A metal in electrostatic equilibrium

• The E-field near the charged surface of the conductor has magnitude $E = \dfrac{\sigma}{\varepsilon_0}$.

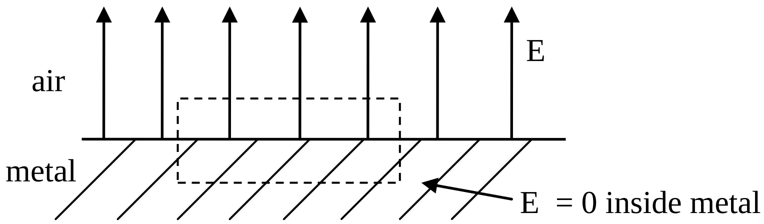
[Note: this is similar, but different than the formula for the field near a plane of charge: $E = \dfrac{\sigma}{2\varepsilon_0}$.]

Proof by Gauss's Law:

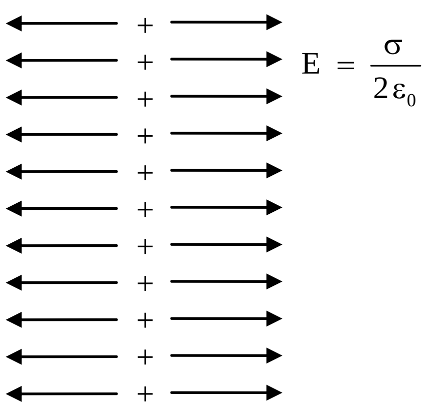


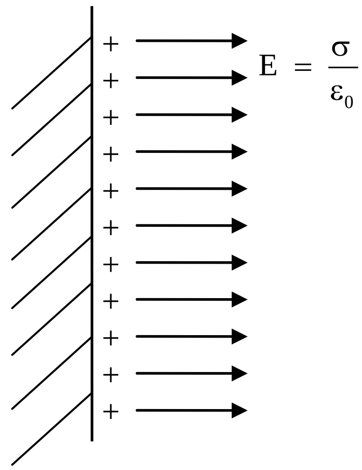
$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0} \quad\Rightarrow\quad E\cancel{A} = \frac{\sigma\cancel{A}}{\varepsilon_0} \quad\Rightarrow\quad E = \frac{\sigma}{\varepsilon_0}$$

Why not $\oint \vec{E}\cdot d\vec{a} = 2EA$? Because E = 0 inside metal

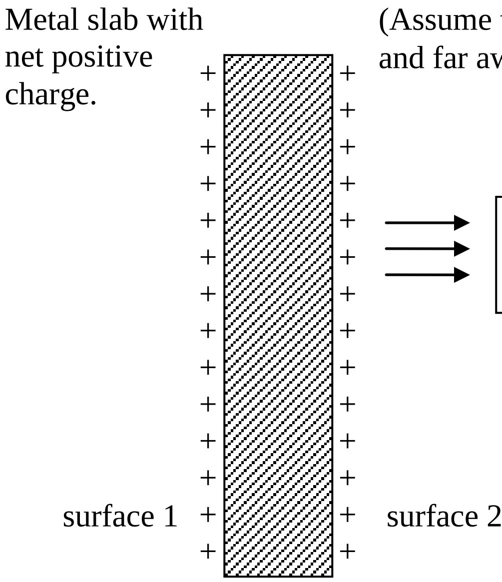


But now, a very puzzling situation has arisen ! We proved before that an infinite plane of charge creates a uniform field $E = \frac{\sigma}{2\varepsilon_0}$. But now we have also proved that a plane of charge on the surface of a metal makes a field $E = \frac{\sigma}{\varepsilon_0}$.





How can both be correct?
The E-field near the surface of a metal is <u>not</u> <u>only</u> due to the charges on that nearby surface. The E-field is always due to <u>all</u> charges, including charges on far-away surfaces!



(Assume that the surfaces on the top and bottom are small and far away so that $\mathbf{E}_{tot}$ is due to surfaces 1 and 2 only.)

$$\boxed{\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}}$$

The charges all arrange themselves so that

- $\mathbf{E} = 0$ inside the metal
- $E = \dfrac{\sigma}{\varepsilon_0}$ just outside metal

