Inductors, Transformers, and AC circuits

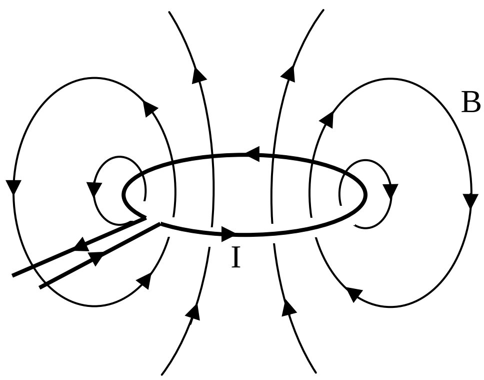
Inductors

An **inductor** is simply a coil of wire. Inductors are used in circuits to store energy in the form of magnetic field energy.

Important point: The magnetic flux Φ_B through any loop is proportional to the current I making the flux. All our formulas for B-field show $B \propto I$:

Biot-Savart: $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{r^2}$ Ampere: $\oint_{\mathcal{L}} \vec{B} \cdot d\vec{\ell} = \mu_0 I_{thru}$

$\Rightarrow \Phi \propto B \propto I \quad \Rightarrow \Phi \propto I$. So the ratio Φ/I is independent of I.



Definition: Self-inductance L of a coil of wire: $\boxed{\Phi_B \equiv L\,I}$

The inductance $L = \Phi_B / I$ is independent of I.

Current I makes B, which makes Φ.

units of inductance [L] = [Φ] / [I] = $T{\cdot}m^2 / A$ = 1 henry (H).

An inductor is a coil of wire. One or a few centimeter-sized loops of wire has L ≅ 1 μH (usually insignificant). A coil with many thousands of turns has L ≅ 1 H (big!).

So, why do we care about inductors? An inductor acts like a "current regulator". An inductor helps to maintain constant current. How is that?

$$\Phi = L \cdot I \quad \Rightarrow \quad \frac{d\Phi}{dt} = L\frac{dI}{dt} \underset{\text{Faraday}}{=} -\mathcal{E}$$

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

⇒ Changing the current in an inductor creates an emf which *opposes* the change in I (by Lenz's Law).

The induced emf is often called a "back emf".

So,

- It is difficult (requires a big external voltage) to change quickly the current in an inductor.
- The current in an inductor <u>cannot</u> change instantly. If it did, there would be an infinite back emf, an infinite E-field to fight the change.

Computing the inductance of a single turn coil (or a few turn coil) is quite messy because the B-field in a loop of wire is non-uniform. The non-uniform B makes computing the magnetic flux $\Phi_B = \int_S \vec{B} \cdot d\vec{a}$ quite difficult. In practice, one determines the inductance of a coil by measuring it, using $\mathcal{E} = -L\frac{dI}{dt}$: put in a known dI/dt, measure emf, compute L.

Computing the inductance of a long solenoid is easy, because the B-field is uniform:

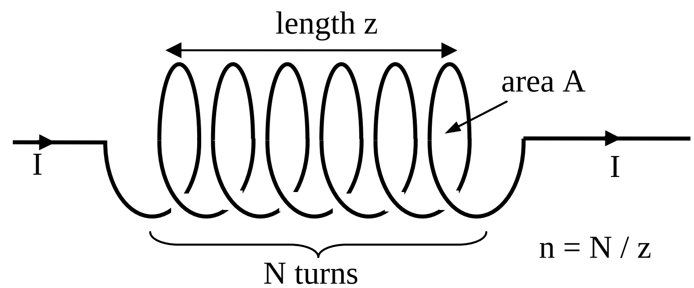
Self-inductance L of a solenoid:

If the coil is very long, $B = \mu_0 n I$ inside ,

so total flux is $\Phi = NBA = N\mu_0 n I A$

inductance $L = \frac{\Phi}{I} = \mu_0 N n A = \mu_0 n^2 A z$

(We use z for length here, to avoid confusion with L for inductance).



Magnetic Energy Density

Recall that for a capacitor, the stored electrostatic potential energy is $U = \frac{1}{2}CV^2$. This energy is in the electric field, and the energy density (energy per volume) is $u_E = \frac{U}{\text{vol.}} = \frac{1}{2}\varepsilon_0 E^2$

For an inductor, the stored energy is $U = \frac{1}{2}LI^2$. This energy is stored in the magnetic field (so we call it magnetostatic potential energy) and the energy density is $u_B = \frac{U}{\text{vol.}} = \frac{1}{2\mu_0}B^2$.

Proof of $U = \frac{1}{2}LI^2$: It takes work to get a current flowing in an inductor. The battery which make the current flow in an inductor must do work against the back emf, which opposes any change in current. Watch closely: power $P = \frac{dU}{dt} = IV = IL\frac{dI}{dt}$, so $dU = P\,dt = IL\,dI$,

so $U = \int dU = \int IL\,dI = L\int I\,dI = \frac{1}{2}LI^2$

Exercise for the motivated student: Show that $u_B = \frac{1}{2\mu_0} B^2$ for the case of a long solenoid.

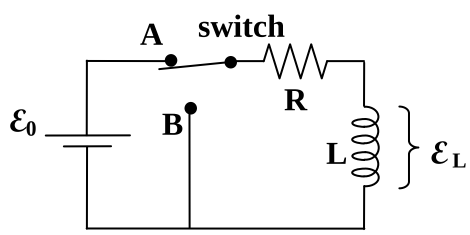
Start with $U = \frac{1}{2} L I^2$, and use the previously found expressions for L and B for a solenoid.

LR circuits (circiuts with L's and R's)

3 things to remember about inductors in circuits:

- An inductor acts like a battery when its current is changing: $\mathcal{E} = -L \frac{dI}{dt}$. The direction of the battery voltage is such as to fight any change in the current.
- The current through an inductor cannot change instantly (because that would cause an infinite $\mathcal{E}$).
- In the steady-state (after a long time), when the current is constant, I = const ⇒ $\mathcal{E}_L = 0$ ⇒ the inductor acts like a short (a zero-resistance wire).

Example: Simple LR circuit.

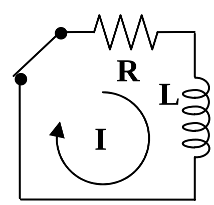


Switch at position A for a long time:

I = constant, so $\mathcal{E}_L = 0$, $I = \mathcal{E}_0 / R$.

At t = 0, switch →B, and the circuit becomes:

The emf in the inductor keeps the current going.



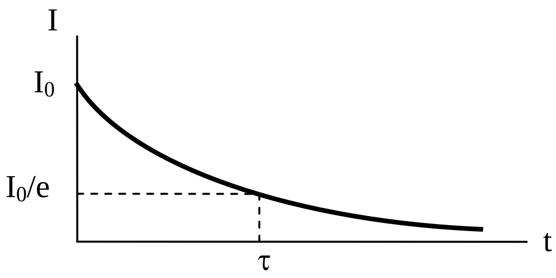
Apply Loop Law: $\mathcal{E}_L = IR = -L \frac{dI}{dt}$

(Note on signs: dI/dt < 0 so $\mathcal{E}_L > 0$.)

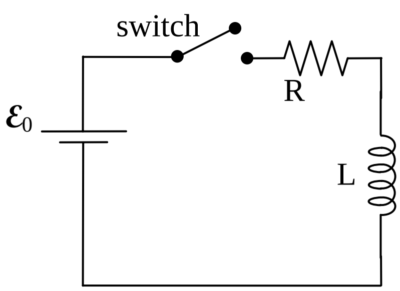
$\frac{dI}{dt} = -\frac{R}{L} I$ This is a differential equation with an expontial solution.

$$I(t) = I_0 e^{-\left(\frac{R}{L}\right)\cdot t} = I_0 e^{-t/(L/R)} = I_0 e^{-t/\tau},$$

$\tau = \frac{L}{R} =$ time constant of LR circuit = time for anything in circuit to change by factor of e



Another LR circuit:



Close switch at t = 0.

At t = 0+, I = 0 (since I cannot change instantly),

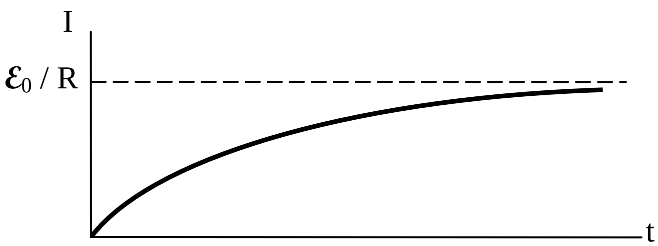
Apply Loop Law, inductor acts like second battery:

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR$$

Initially, I = 0, so $\mathcal{E}_0 - L\frac{dI}{dt} = 0$, $\left.\frac{dI}{dt}\right|_{t=0} = \frac{\mathcal{E}_0}{L}$

As t $\uparrow$, I $\uparrow$, $V_R = I R \uparrow$, $|\mathcal{E}_L| = |L\, dI / dt|$

As t $\rightarrow \infty$, $|\mathcal{E}_L| \rightarrow 0$, $\mathcal{E}_0 = IR$



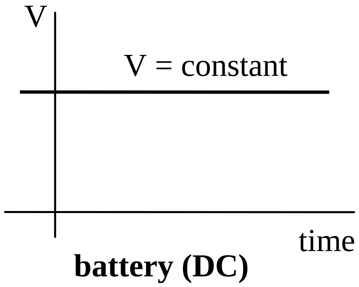
$$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-t/(L/R)}\right]$$

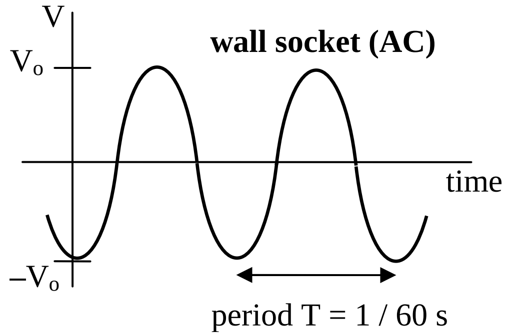
AC Voltage and Current

Batteries produce voltage that is constant in time, DC voltage.

The wall socket produces sinusoidally-varying voltage, AC voltage.

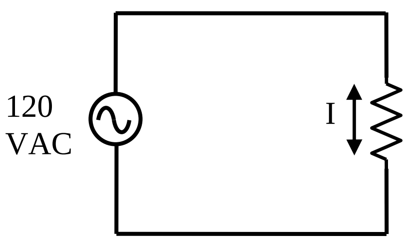
(DC originally stood for "direct current" but now it just means "constant in time". AC is short for "alternating current" but now means "sinusoidally-varying".)





Wall socket voltage: $V = V(t) = V_o \sin\left(2\pi \frac{t}{T}\right) = V_o \sin(2\pi f\, t) = V_o \sin(\omega t)$

In the US, the frequency of "line voltage" is f = 60 Hz = 60 cycles per second (Recall f = 1 / T, period T = 1/60 s)



symbol for AC voltage

AC voltage causes AC current in resistor. Current actually flows back and forth, 60 times a second.

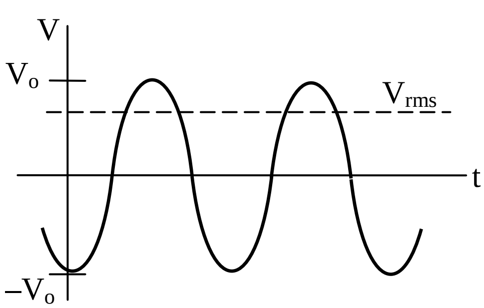
$$I = \frac{V}{R} = \frac{V_0}{R}\sin(\omega t) = I_o \sin(\omega t)$$

The instantaneous voltage is (+) as often as (–), so $\overline{V} = V_{avg} = 0$, but $|V|_{avg} \neq 0$.

Electrical engineers <u>always</u> report AC voltage using a kind of average called "root-mean-square" or rms average.

$$\text{VAC} = \text{"volts AC"} = V_{rms} = \sqrt{\overline{V^2}} = 120\,\text{V (in US)}$$

The average voltage V_{rms} is less than the peak voltage V_o by a factor of $\sqrt{2}$: $V_{rms} = \frac{V_o}{\sqrt{2}}$



Why $\sqrt{2}$? $V \propto \sin(\omega t)$, $V^2 \propto \sin^2(\omega t)$

sin varies from +1 to –1 (sin = +1 → 0 → –1 → 0 → +1→ 0 → –1 → 0 → ..)

$\sin^2$ varies from 0 to +1 ($\sin^2 = +1 \rightarrow 0 \rightarrow +1 \rightarrow 0 \rightarrow +1 \rightarrow 0 \rightarrow +1 \rightarrow 0 \rightarrow ..$)

The average of $\sin^2$ is ½ .

$$V_{rms} = \sqrt{\overline{V^2}} = \sqrt{\overline{V_o^2 \sin^2(\omega t)}} = V_o \sqrt{\overline{\sin^2(\omega t)}} = V_o \sqrt{\frac{1}{2}} = \frac{V_o}{\sqrt{2}}$$

Wall socket voltage or "line voltage" : $V_{rms} = 120$ V , $V_{peak} = V_0 = \sqrt{2}\, V_{rms} \simeq 170\,\text{V}$

Average vs. instantaneous quantities:

Power $P = IV \overset{(AC)}{=} I_o \sin(\omega t) V_o \sin(\omega t) = I_o V_o \sin^2(\omega t)$

Since $\sin^2$ alternates between 0 and +1, the power P alternates between 0 and $P_{max} = I_o V_o$.

The average value of $\sin^2$ = ½ $\Rightarrow$ $P_{avg} = I_o V_o \cdot \frac{1}{2} = \frac{I_o}{\sqrt{2}} \cdot \frac{V_o}{\sqrt{2}} = I_{rms} V_{rms}$

The old formula P = IV works OK with AC quantities if we use P_{avg} , I_{rms} , and V_{rms}.

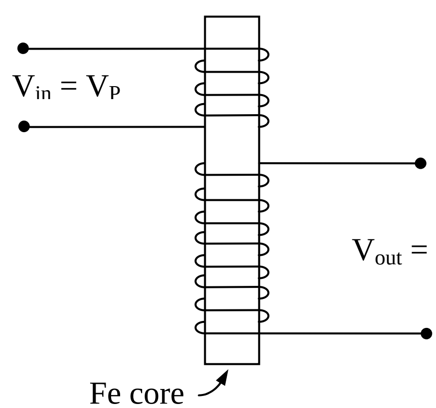
All the old DC formulas, V = IR , $P = IV = V^2/R = I^2 R$, still work fine for AC if we use I_{rms}, V_{rms}, and $P_{avg.}$

Transformers

The entire electrical power distribution system in the civilized world depends on a simple device called a transformer. A transformer is a device for transforming AC voltage from one value (say 120 VAC) to another value (like 10 VAC or 2000 VAC). A transformer is made of 2 coils of wire, usually wrapped around an iron core. It is a simple device with no moving parts.

Primary coil = input coil, with N_P turns

Secondary coil = output coil, with N_S turns



We will show below that $V_{out} = V_S = \frac{N_S}{N_P} \cdot V_P$ or $\boxed{\frac{V_S}{V_P} = \frac{N_S}{N_P}}$

This "Transformer Equation" says that, for AC voltage, the voltage ratio is equal to the turns ratio.

NOTE: the transformer only works for AC voltage. If V_{in} is DC, then $V_{out} = 0$.

$\frac{N_S}{N_P} = \frac{V_S}{V_P} > 1 \quad \Rightarrow$ "step-up transformer"

$\frac{N_S}{N_P} = \frac{V_S}{V_P} < 1 \quad \Rightarrow$ "step-down transformer"

(A step-down transformer gives a smaller V, but a larger current I.)

Transformers work because of Faraday's Law:

V_P (AC) $\Rightarrow$ I_P (AC) $\Rightarrow$ B_P (AC) $\Rightarrow$ B_S (AC) + Faraday $\Rightarrow$ $\mathcal{E} = V_S$

Proof of the Transformer Equation:

If we apply Faraday's Law to the primary and secondary coils, we get:

(1) $V_S = N_S \frac{d\Phi}{dt}$

(2) $V_P = N_P \frac{d\Phi}{dt}$

same $\Phi = B\,A$ in each turn of primary and secondary because the iron core "guides flux" from P to S.

(1) ÷ (2) $\Rightarrow \frac{V_S}{V_P} = \frac{N_S}{N_P}$ (End of proof.)

If a transformer is well-designed, only 1 to 5% power in is lost to heating of coils and eddy currents in the iron core.

$\Rightarrow \quad P_{out} \cong P_{in} \quad \Rightarrow \quad I_S V_S = I_P V_P \quad \Rightarrow \quad \frac{I_S}{I_P} = \frac{V_P}{V_S} = \frac{N_P}{N_S}$

A step-down transformer produces a smaller voltage, but a bigger current (same P = I V).

Light bulbs and appliances with motors (vacuum cleaners, blenders) use AC voltage to operate. But devices with electronic circuits (TV's, computers, phones, etc) need DC voltage to function. The "power supply" in computers and TV's converts the AC voltage from the wall socket into DC voltage (usually 10-15 V) that the electronic circuitry needs.

Example of use of transformers: Suppose you want to melt a nail by putting a big current through it. What happens if you try to melt the nail by putting 120 VAC (from your wall socket) across the nail? Answer: you will blow a fuse or trip a breaker. The resistance of a nail is quite small: $R_{nail} \cong 10^{-3}\,\Omega$. The current produced by a 120 V voltage difference across the nail is huge: $I_{nail} = \dfrac{V}{R_{nail}} = \dfrac{120\,\text{V}}{10^{-3}\,\Omega} = 120000\text{ A}$. This will never happen since your breaker will trip when the current exceeds 15 A. (Here's an experiment you should never try at home: Bend a nail into a U shape and plug it into your wall socket. Watch the lights go out.)

So how do we melt that nail? Solution: Use a 100-to-1 step-down transformer.

$$\frac{N_S}{N_P} = \frac{1}{100} = \frac{V_S}{V_P}\,, \qquad V_P = V_{in} = 120\text{ VAC}$$

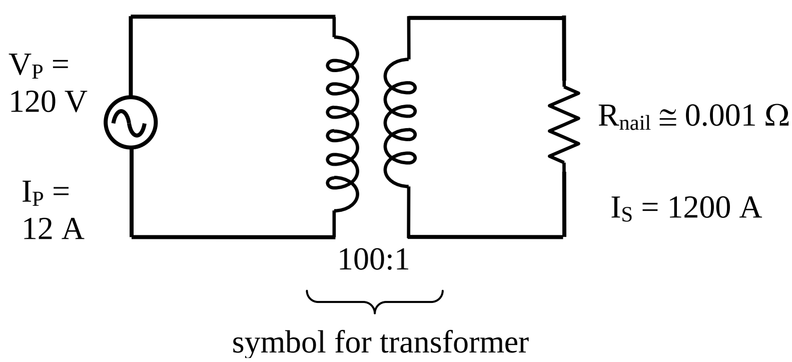
$$V_S = V_{out} = \frac{N_S}{N_P}\cdot V_P = \frac{1}{100}\cdot 120\text{ V} = 1.2\text{ VAC}$$

$$I_S = I_{out} = \frac{V_S}{R_{nail}} = \frac{1.2\,\text{V}}{10^{-3}\,\Omega} = 1200\text{ A} \quad \text{(enough to melt the nail)}$$

How much current will this draw from the wall socket? Recall that $P_{in} = P_{out}$ or $I_S V_S = I_P V_P$.

$$I_P = I_{in} = I_S\cdot\frac{V_S}{V_P} = I_S\cdot\frac{N_S}{N_P} = (1200\text{ A})\left(\frac{1}{100}\right) = 12\,\text{A} \quad \text{(Not enough to blow the fuse.)}$$

Circuit diagram:

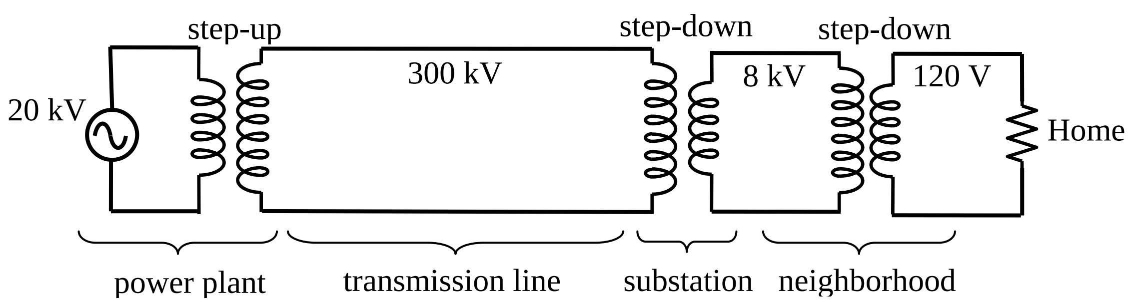


Power dissipated in nail = $I_S^2 R = (1200)^2 (10^{-3}) = 1440$ W $\Rightarrow$ will melt nail.

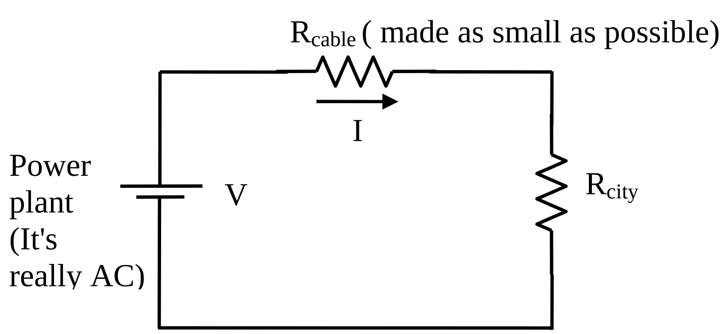
Transformers and Power Distribution

Economical power distribution is only possible because of transformers. Electrical power is transmitted from the power plant to the city by big aluminum cables (power lines). Some energy is inevitably wasted because the power lines have a resistance, and so they get hot: $P_{lost} = I^2 R_{cable}$. In order to minimize this waste, the power must be transmitted from the plant to the city at very high voltage (typically 300 kV). A high voltage allows a small current, at a given power (since P = IV). And a small current means small I^2R losses in the cable.

When the high-voltage power is delivered to the city, step-down transformers are used to transform the very dangerous high voltage down to the not-so-dangerous 120 V before it enters your home. The voltage is stepped down in stages as it is distributed throughout the city.



A very simplified model of power distribution (transformers not shown):



Some typical numbers:

Power output of plant = P_{out} = 100 MW to 1 GW = 10^7 to 10^8 W (fixed by demands of the city)

$P_{out} = I\,V$, $P_{lost} = I^2 R_{cable}$ Using $I = P_{out} / V$, we get $P_{lost} = \frac{P_{out}^{\;2}}{V^2} \cdot R_{cable}$

Fraction of power wasted = $\frac{P_{lost}}{P_{out}} = \frac{P_{out}}{V^2} \cdot R_{cable}$

If $R_{cable} \approx 10\Omega$ and $P_{out} = 10^8$ W, then

If V = 50,000 V : $\frac{P_{lost}}{P_{out}} = \frac{10^8}{\left(5\times10^4\right)^2} \cdot 10 = 0.4$ (40% lost!)

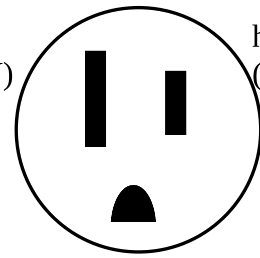
If V = 200,000 V : $\frac{P_{lost}}{P_{out}} = \frac{10^8}{\left(2\times10^5\right)^2} \cdot 10 = 0.025$ (2.5% lost)

Boosting the voltage at which the power is transmitted makes the losses acceptably small.

Household Wiring

Wall socket = 3-prong plug

The short slot is the dangerous high-voltage one; short slot is harder to stick your finger in.

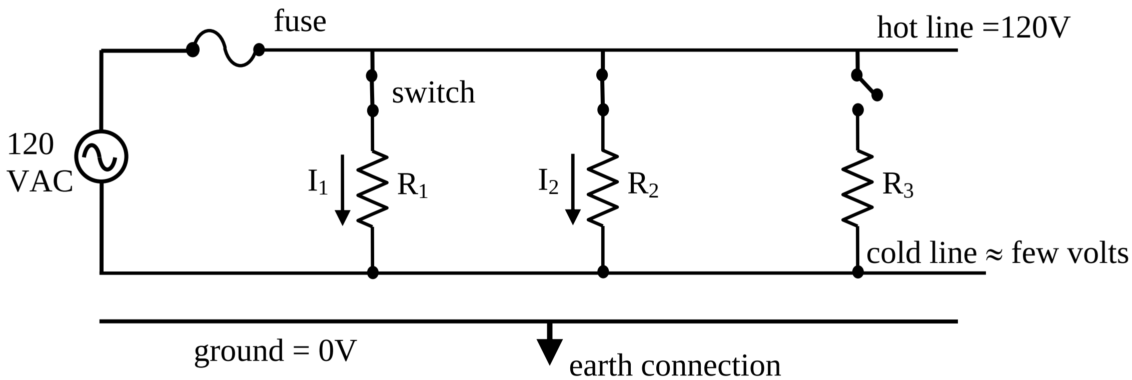


Standard electrical wiring colors:

- black = hot (120 V) "charred black"
- white = cold (few V) "white ice cold"
- green = ground (0 V) "green grass"

Never assume the wiring colors are correct! Always check with a voltmeter.

The resistance of copper wires in the walls of your home is less than 0.1 Ω. So $R_{wire} << R_{bulb} \approx$ 100 Ω. R_{wire} is small, but not zero ⇒ wires get hot if too much current ⇒ fire hazard. So all circuits in your house have fuses or circuit breakers which automatically break the circuit if the current exceeds 15 A.



Example of voltage drop along a wire: What is the resistance of copper wire, length L = 10 m, diameter = 1 mm, $\rho = 1.7 \times 10^{-8}\ \Omega\cdot\text{m}$ (typical of wires in the walls of your house.)

$$R_{wire} = \frac{\rho L}{A} = \frac{\rho L}{\pi r^2} = \frac{(1.7\times10^{-8})(10)}{\pi(0.001)^2} = 0.054\,\Omega$$

If the current through this wire is I = 15 A (close to tripping the breaker), what is the voltage drop along this wire? $V_{wire} = I\,R_{wire} = (15\text{ A})(0.054\ \Omega) = 0.81\text{ V}$

Cost of electricity

Power company charges for total energy used. energy = power × time (P = W / t, W = P t)

Unit of energy = kilowatt-hour (kW·h) = two hairdryers on for 1 hour.

1 kW·h costs about 10 cents (varies).

Example of energy cost. What's the bill for a 500 W hairdryer left on for 1 year?

$$1\,\text{year}\cdot 365\frac{\text{d}}{\text{y}}\cdot 24\frac{\text{h}}{\text{d}} = 8760\,\text{hours}\ ,\quad \frac{\$0.10}{\text{kW}\cdot\text{h}}\cdot 8760\,\text{h}\cdot 0.5\,\text{kW} = \$438\ \ \text{(yikes!)}$$