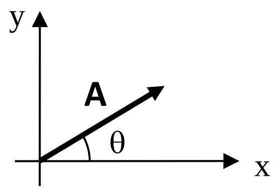
Vector Components

Vectors such as displacement are "real" things, but they are very difficult to describe without first defining a coordinate system. If I say I walk *2 m* at an angle of *30 degrees*, it is a fairly meaningless statement unless *30 degrees* is relative to some reference. Once we have defined a coordinate system we can discuss the direction of a vector in precise terms. As we have mentioned before…

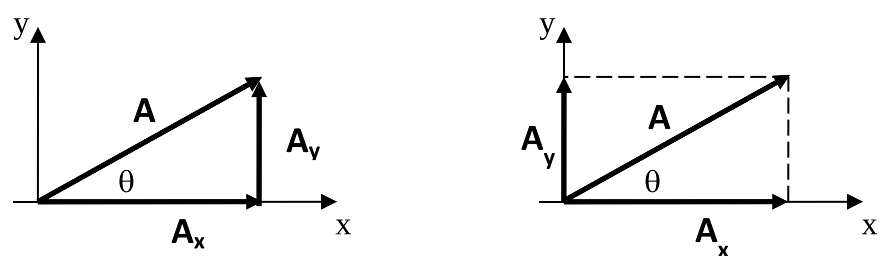
A key step in vector problems is defining a coordinate system

Once we have defined a coordinate system, we can clearly define a vector. In two dimensions (2-d), we need two pieces of information to define a vector, its magnitude and the angle it makes with the coordinate system.



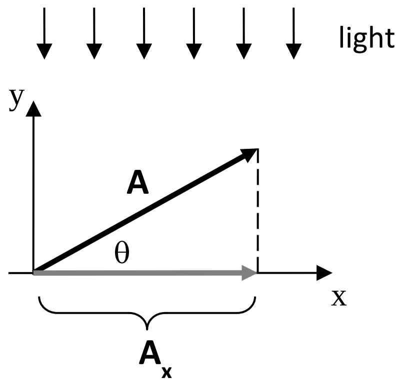
An extremely powerful method of working with vectors is to break vectors into ***components***. In two dimensions (2-d) we can make any arbitrary vector by adding together one vector that lies purely in the horizontal direction to another vector that lies purely in the vertical direction. By convention, we usually refer to the horizontal direction as the "*x*" direction, and we usually refer to the vertical direction as the "*y*" direction. If we use the example of displacement, to get to some final position, we could walk "as the crow flies" in a single straight line at the necessary angle. Or instead, we could choose to walk purely along the *x*-direction and purely along the *y*-direction as shown in the example below.

$$\vec{A} = \vec{A}_x + \vec{A}_y$$



In this way, we can decompose any vector into an ***x-component*** and ***y-component***. These vector components can be thought of as the 1-d vectors we encountered earlier. We can define each component by a combination of its magnitude and a plus or minus sign. For instance, since an *x*-component will only lie along the *x*-direction, its directional information can be given by just a plus or minus sign. Of course, if a vector lies purely in one direction, one of its components will be zero.

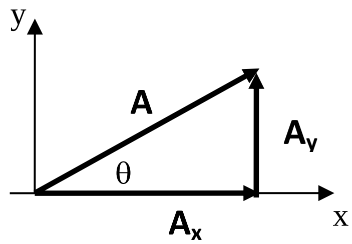
Another way to think of vector components is to imagine the "shadow" or "projection" a vector would cast onto one of the coordinate axes. For example, the shadow vector $\vec{A}$ casts onto the *x*-axis by a distant light source directly "overhead" would be its *x*-component.



Notice that we tend to use the convention of identifying vector components with subscripts, such as in the example above where $\vec{A}_x$ represents the *x*-component of $\vec{A}$.

Since we use coordinate systems in which the *x*-axis and *y*-axis are perpendicular, the x-component and y-component of a vector will always be at a 90 degree angle to each other. Conveniently, this fact will allow us to use the rules of trigonometry for right triangles to calculate the components of a vector.

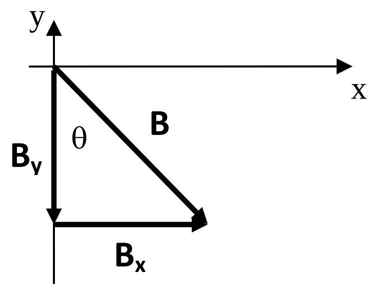
For the vector shown above we could use the rules of trigonometry, and the Pythagorean Theorem to write that



$$\vec{A}_x = +A\cos\theta \qquad \vec{A}_y = +A\sin\theta \qquad |\vec{A}| = \left(A_x^2 + A_y^2\right)^{\frac{1}{2}}$$

Remember that when we remove the little "arrow" symbol we are representing only the magnitude. In words, the *x*-component in this example has a magnitude that is equal to the magnitude of the vector times the cosine of the angle given. The *x*-component in this example points in the positive direction.

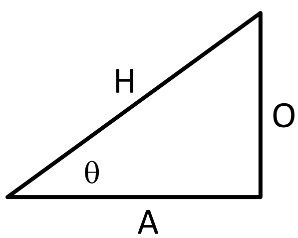
Do NOT assume the *x*-component always uses $\cos\theta$ and the *y*-component always uses $\sin\theta$, it depends on the geometry!



Notice that in the diagram above for vector $\vec{B}$ and the given angle

$$\vec{B}_x = +B\sin\theta \qquad \vec{B}_y = -B\cos\theta \qquad |\vec{B}| = \left(B_x^2 + B_y^2\right)^{\frac{1}{2}}$$

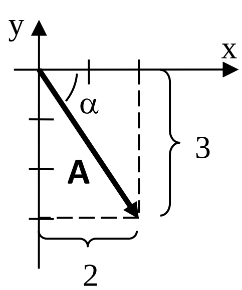
Where when we remove the "arrow" symbol from the vector we mean just the magnitude. A good question to always ask when calculating the vector components is whether you are opposite or adjacent to the given angle for the component of interest, and remember SOHCAHTOA. For right triangles:



$$\sin\theta = \frac{O}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{O}{A}$$

Example of vector math:

Suppose $\vec{A}$ has components $\vec{A}_x = +2, \vec{A}_y = -3$
What is the magnitude of this vector and what angle does it make with the *x*-axis?



$$|\vec{A}| = \left(A_x^2 + A_y^2\right)^{\frac{1}{2}}$$
$$= ([2]^2 + [-3]^2)^{\frac{1}{2}}$$
$$= (4+9)^{\frac{1}{2}}$$
$$= \sqrt{13} \cong 3.6$$

$$\tan\alpha = \frac{3}{2} \;\rightarrow\; |\alpha| = 56.3°$$

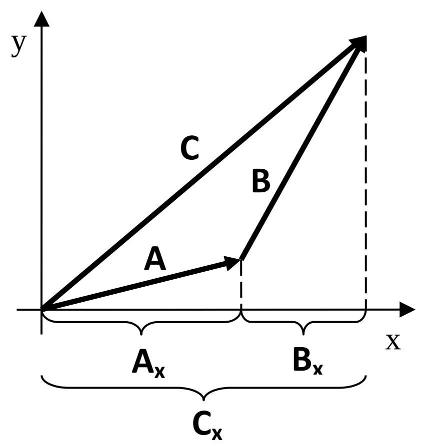
Generally we refer to angles as growing positively as we rotate counterclockwise from the *x*-axis, so in the example above, we would usually say this vector makes an angle of $-56.3°$ with the *x*-axis (notice the negative sign).

Vector Arithmetic with Components

Vector equations using 2-d vectors can be thought of as shorthand for representing two component equations. For example, if we add vector $\vec{A}$ to vector $\vec{B}$ to get vector $\vec{C}$, we have seen how we can visualize this process graphically. However, we can also just add the *x*-components and separately add the *y*-components to get the final result.

Again, displacement is a useful conceptual example. To get from the origin to the final position we could walk just the *x*-components and then walk the *y*-components.

Vector Addition by Components:



$$\vec{C} = \vec{A} + \vec{B}$$
$$\vec{C}_x = \vec{A}_x + \vec{B}_x$$
$$\vec{C}_y = \vec{A}_y + \vec{B}_y$$
$$\left|\vec{C}\right| = \left(C_x^2 + C_y^2\right)^{\frac{1}{2}}$$

To perform vector addition with components we follow these steps:

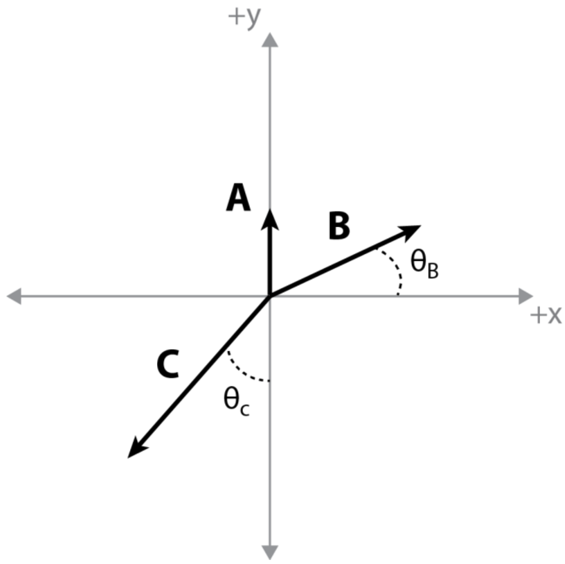
1. Define a coordinate system (or it is defined for us)
2. Draw the vectors in the coordinate system (or they are drawn for us)
3. Break the vectors into components using the appropriate trigonometric relationships
4. Add the components in the *x*-direction, and add the components in the *y*-direction
5. Knowing the result of the component additions, we can find the magnitude of the final vector and its direction using the Pythagorean Theorem and trigonometric relationships

Note that when we say “add” the components, one of the components may be in the negative direction. Furthermore if our vector equation involves subtracting two vectors from each other we must take into account the operation of subtraction. Personally, I always like to think about subtraction as “addition of a negative” and work out problems with that approach.

Example of Vector Addition with Components

Given the following vectors and angles, what is magnitude and direction of the vector given by the following equation?

$$\vec{A} - \vec{B} + \vec{C}$$



$$|\vec{A}| = 2 \text{ , points along y-axis}$$

$$|\vec{B}| = 3 \quad , \quad \theta_B = 30°$$

$$|\vec{C}| = 4 \quad , \quad \theta_c = 45°$$

Notice that the vector equation will define a new vector. First we will find all of the components of the 3 vectors involved in the equation.

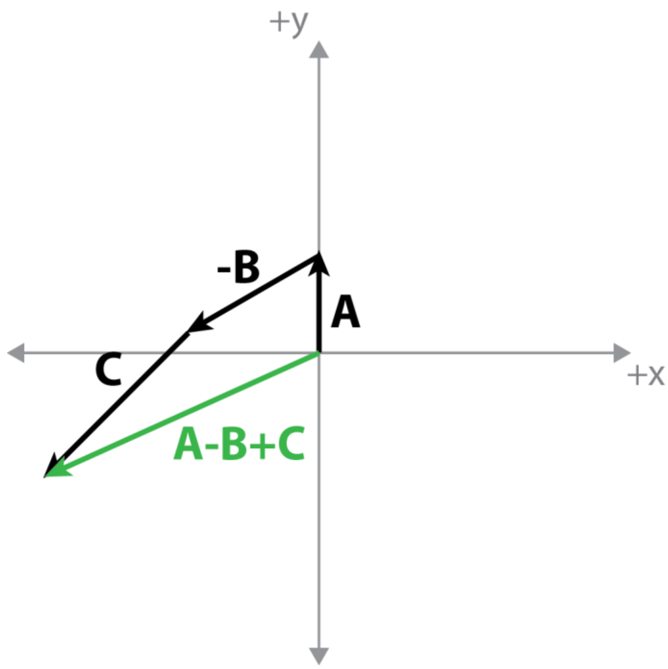
$$\vec{A}_x = 0 \ , \ \vec{A}_y = +2$$
$$\vec{B}_x = +|\vec{B}| \cos\theta_B \quad , \quad \vec{B}_y = +|\vec{B}| \sin\theta_B$$
$$\vec{C}_x = -|\vec{C}| \sin\theta_C \quad , \quad \vec{C}_y = -|\vec{C}| \cos\theta_C$$

Now we can calculate all of these values. Notice we already knew the components of vector $\vec{A}$ since it is only pointing purely along the y-axis

$$(\vec{A}_x = 0 \, , \, \vec{A}_y = +2) \quad (\vec{B}_x = +2.60 \, , \, \vec{B}_y = +1.50) \quad (\vec{C}_x = -2.83 \, , \, \vec{C}_y = -2.83)$$

Plugging into our vector formula we have

$$\vec{A}_x - \vec{B}_x + \vec{C}_x = 0 - (+2.60) + (-2.83) = -5.43$$
$$\vec{A}_y - \vec{B}_y + \vec{C}_y = 2 - (+1.50) + (-2.83) = -2.33$$
$$|\vec{A} - \vec{B} + \vec{C}| = ([-5.43]^2 + [-2.33]^2)^{\frac{1}{2}} = 5.91$$



Notice that our component calculations gave us a total vector that is pointed down and to the left relative to the coordinate system we had originally chosen. When we graphically add the vectors the result agrees with the component calculations. Finally to get the direction of the result we can use the inverse tangent function. Since I know the quadrant in which the final vector points, I usually just use the magnitude of the components to find the angle between 0 and 90 degrees relative to the x axis.

$$\tan^{-1}\left(\frac{2.33}{5.43}\right) = 23.22°$$

So the absolute angle relative to the x-axis is either written as 180 degrees plus this result, or as a negative angle

$$final\ angle = 203.22°\ or\ equivanlently, -156.78°$$

As a final note, we could also have a vector equation where one of the vectors is multiplied by a constant other than 1 or -1. In those cases, we would follow this same process, but scale the components by the constant. So for the example we just did, if we instead had the vector equation

$$\vec{A} - \vec{B} + 2\vec{C}$$

We would have calculated

$$\vec{A}_x - \vec{B}_x + 2\vec{C}_x = 0 - (+2.60) + 2(-2.83) = -8.26$$
$$\vec{A}_y - \vec{B}_y + 2\vec{C}_y = 2 - (+1.50) + 2(-2.83) = -5.16$$

