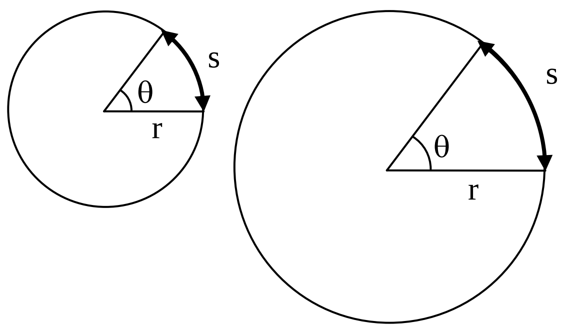
Rotational Motion

We are going to consider the motion of a rigid body about a fixed axis of rotation.

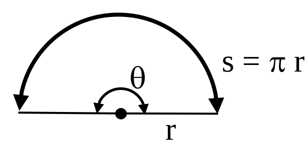
The angle of rotation is measured in radians: $\boxed{\theta(\text{rads}) \equiv \frac{s}{r} \text{ (dimensionless)}}$



Notice that for a given angle θ, the ratio s/r is independent of the size of the circle.

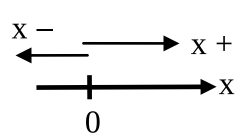
Example: How many radians in 180°? Circumference C = 2 π r

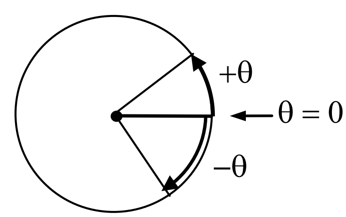
$\theta = \frac{s}{r} = \frac{\pi r}{r} = \pi$ rads π rads = 180°, 1 rad = 57.3°



Angle θ of a rigid object is measured relative to some reference orientation, just like 1D position x is measured relative to some reference position (the origin).

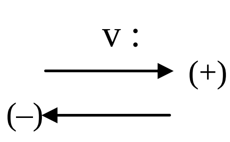
Angle θ is the "rotational position". Like position x in 1D, rotational position θ has a sign convention. Positive angles are CCW (counter-clockwise).

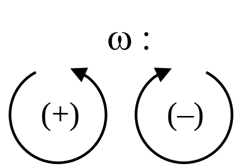




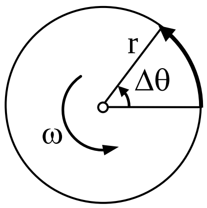
Definition of *angular velocity*: $\boxed{\omega \equiv \frac{\Delta\theta}{\Delta t} \text{ (rad/s)}}$ (like $v \equiv \frac{\Delta x}{\Delta t}$)

In 1D, velocity v has a sign (+ or –) depending on direction. Likewise ω has a sign convention, depending on the sense of rotation.





For rotational motion, there is a relation between *tangential velocity* v (velocity along the rim) and angular velocity θ.



Δs in time Δt

$$\Delta\theta = \frac{\Delta s}{r} \quad \Rightarrow \quad \Delta s = r\,\Delta\theta \ ,$$

$$v = \frac{\Delta s}{\Delta t} = \frac{r\,\Delta\theta}{\Delta t} = r\,\omega$$

$$\boxed{v = r\,\omega}$$

Definition of *angular acceleration* : $\boxed{\alpha \equiv \frac{\Delta\omega}{\Delta t} \ (\text{rad/s}^2)}$ $\qquad$ (like $a \equiv \frac{\Delta v}{\Delta t}$)

α = rate at which ω is changing.
ω = constant $\Leftrightarrow$ $\alpha = 0$ $\Rightarrow$ speed v along rim = constant = $r\,\omega$

Equations for constant α:

Recall from Chapter 2: We defined $v = \frac{\Delta x}{\Delta t}$, $a = \frac{\Delta v}{\Delta t}$,

and then showed that, if a = constant, $\begin{cases} v = v_0 + a\,t \\ x = x_0 + v_0 t + \frac{1}{2} a\,t^2 \\ v^2 = v_0^2 + 2a(x - x_0) \end{cases}$

Now, in Chapter 8, we define $\omega = \frac{\Delta\theta}{\Delta t}$, $\alpha = \frac{\Delta\omega}{\Delta t}$.

So, if α = constant, $\begin{cases} \omega = \omega_0 + \alpha\,t \\ \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha\,t^2 \\ \omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \end{cases}$

Same equations, just different symbols.

Example: Fast spinning wheel with ω_0 = 50 rad/s (about 8 rev/s). Apply brake and wheel slows at α = –10 rad/s. How many revolutions before the wheel stops?

Use $\omega^2 = \omega_0^2 + 2\alpha\ \Delta\theta$, $\omega_{\text{final}} = 0 \Rightarrow\ 0 = \omega_0^2 + 2\alpha\ \Delta\theta \ \Rightarrow\ \Delta\theta = -\frac{\omega_0^2}{2\alpha} = -\frac{50^2}{2(-10)} = 125 \text{ rad}$

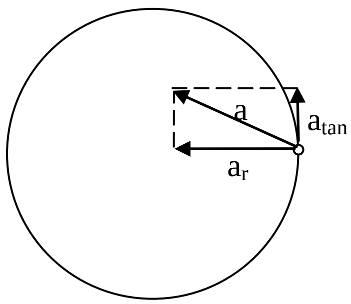
$$125 \text{ rad} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} = 19.9 \text{ rev}$$

Definition of *tangential acceleration* a_{tan} = rate at which speed v along rim is changing

$$a_{tan} \equiv \frac{\Delta v}{\Delta t} = \frac{\Delta(r\,\omega)}{\Delta t} = \frac{r\,\Delta\omega}{\Delta t} \quad \Rightarrow \quad \boxed{a_{tan} = r\,\alpha}$$

a_{tan} is different than the radial or centripetal acceleration $a_r = \dfrac{v^2}{r}$

a_r is due to change in *direction* of velocity **v**
a_{tan} is due to change in *magnitude* of velocity, speed v



a_{tan} and a_r are the tangential and radial components of the acceleration vector **a**.

$$|\vec{a}| = a = \sqrt{a_{tan}{}^2 + a_r{}^2}$$

Angular velocity ω also sometimes called angular frequency.
Difference between angular velocity ω and frequency f:

$$\omega = \frac{\#\text{ radians}}{\text{sec}}, \qquad f = \frac{\#\text{ revolutions}}{\text{sec}}$$

T = period = time for one complete revolution (or cycle or rev) ⇒

$$\omega = \frac{2\pi\text{ rad}}{T} = \frac{2\pi}{T}, \qquad f = \frac{1\text{ rev}}{T} = \frac{1}{T} \quad \Rightarrow \quad \boxed{\omega = 2\,\pi\,f}$$

Units of frequency f = rev/s = hertz (Hz) . Units of angular velocity = rad /s = s^{-1}

Example: An old vinyl record disk with radius r = 6 in = 15.2 cm is spinning at 33.3 rpm (revolutions per minute).

- What is the period T? $\dfrac{33.3\text{ rev}}{1\text{min}} = \dfrac{33.3\text{ rev}}{60\text{ s}} \Rightarrow \dfrac{60\text{ s}}{33.3\text{ rev}} = \dfrac{(60/33.3)\text{ s}}{1\text{rev}} \simeq 1.80$ s/rev

  ⇒ period T = 1.80 s

- What is the frequency f ? f = 1 / T = 1 rev / (1.80 s) = 0.555 Hz
- What is the angular velocity ω ? $\omega = 2\,\pi\,f = 2\,\pi\,(0.555\text{ s}^{-1}) \simeq 3.49\text{ rad/s}$
- What is the speed v of a bug hanging on to the rim of the disk?

$v = r\,w = (15.2\text{ cm})(3.49\text{ s}^{-1}) = 53.0\text{ cm/s}$

• What is the angular acceleration α of the bug? $\alpha = 0$, since ω = constant

• What is the magnitude of the acceleration of the bug? The acceleration has only a radial component a_r , since the tangential acceleration $a_{tan} = r\,\alpha = 0$.

$$a = a_r = \frac{v^2}{r} = \frac{(0.530\,\text{m/s})^2}{0.152\,\text{m}} = 1.84\ \text{m/s}^2 \quad \text{(about 0.2 g's)}$$

As we shall see, for every quantity in linear (translational) motion, there is an analogous quantity in rotational motion:

Translation	$\leftrightarrow$	**Rotation**
x	$\leftrightarrow$	θ
$v = \dfrac{\Delta x}{\Delta t}$	$\leftrightarrow$	$\omega = \dfrac{\Delta \theta}{\Delta t}$
$a = \dfrac{\Delta v}{\Delta t}$	$\leftrightarrow$	$\alpha = \dfrac{\Delta \omega}{\Delta t}$
F	$\leftrightarrow$	(?)
M	$\leftrightarrow$	(?)
F = Ma	$\leftrightarrow$	(?) = (?) α

The rotational analogue of force is *torque*.

Force F causes acceleration a $\leftrightarrow$ torque τ causes angular acceleration α

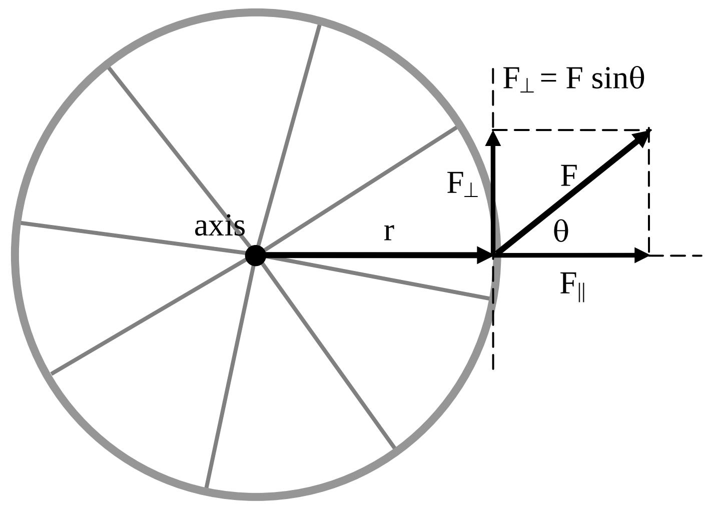
The torque (pronounced "tork") is a kind of "rotational force".

magnitude of torque: $\boxed{|\tau| \equiv r \cdot F_\perp}$ $\qquad [\tau] = [r][F] = \text{m N}$

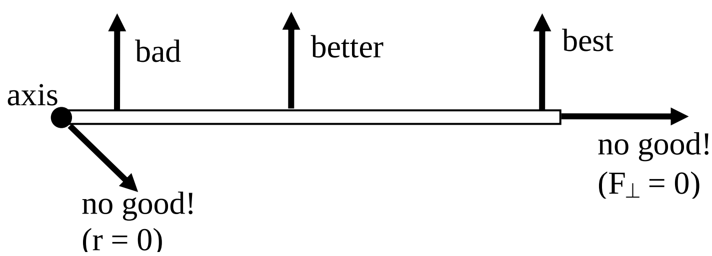
r = "lever arm" = distance from axis of rotation to point of application of force

$F_\perp$ = component of force perpendicular to lever arm

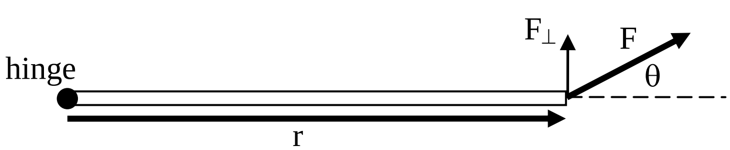
Example: Wheel on a fixed axis: Notice that only the perpendicular component of the force **F** will rotate the wheel. The component of the force parallel to the lever arm ($F_{||}$) has no effect on the rotation of the wheel.



If you want to easily rotate an object about an axis, you want a large lever arm r and a large perpendicular force $F_\perp$:



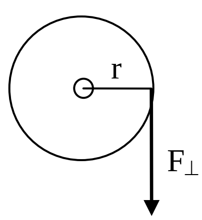
Example: Pull on a door handle a distance r = 0.8 m from the hinge with a force of magnitude F = 20 N at an angle $\theta = 30^o$ from the plane of the door, like so:



$\tau = r\,F_\perp = r\,F \sin\theta =$

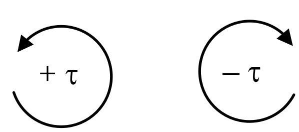
$(0.8 \text{ m})(20 \text{ N})(\sin 30^o) = 8.0 \text{ m}\cdot\text{N}$

Another example: a Pulley



$\tau = r\,F_\perp$

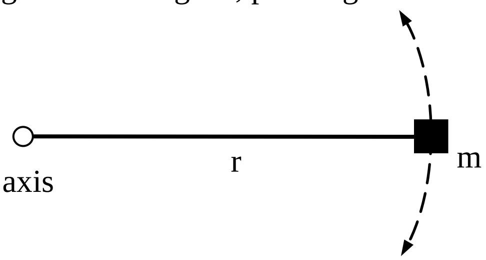
Torque has a sign (+ or –) :
Positive torque causes counter-clockwise CCW rotation.
Negative torque causes clockwise (CW) rotation.



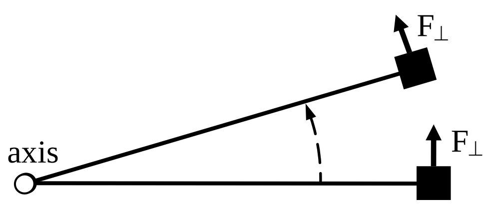
If several torques are applied, the net torque causes angular acceleration.

$\tau_{net} = \sum \tau \propto \alpha$

To see the relation between torque τ and angular acceleration α , consider a mass *m* at the end of light rod of length *r*, pivoting on an axis like so:



Apply a force $F_\perp$ to the mass, keeping the force perpendicular to the lever arm r.



acceleration $a_{tan} = r\,\alpha$

Apply $F_{net} = m\,a$, along the tangential direction:

$F_\perp = m\,a_{tan} = m\,r\,\alpha$

Multiply both sides by r (to get torque in the game): $r\,F_\perp = (m\,r^2)\,\alpha$

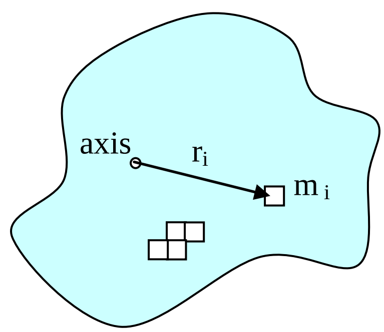
Define "moment of inertia" = $I = m\,r^2$

$\Rightarrow$ $\boxed{\tau = I \cdot \alpha}$ (like $F = m \cdot a$)

Can generalize definition of I:

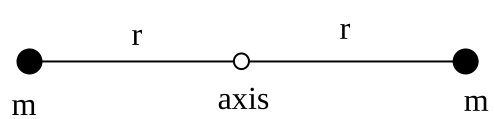
Definition of *moment of inertia* of an extended object about an axis of rotation:

$$\boxed{I \equiv \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \ldots}$$



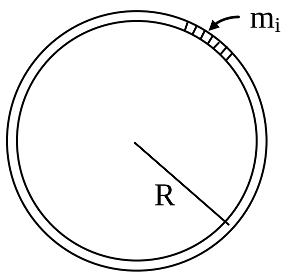
Examples:

• 2 small masses on rods of length r:



$I = 2\,m\,r^2$

• A hoop of total mass M, radius R, with axis through the center, has $I_{hoop} = M\,R^2$



$$I = \sum_i m_i r_i^2 = \left(\sum_i m_i\right) R^2 = M R^2$$

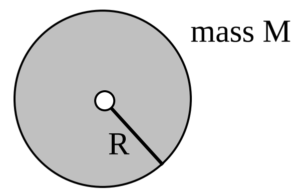
(since $r_i = R$ for all i)

In detail:

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots = m_1 R^2 + m_2 R^2 + m_3 R^2 + \ldots$$
$$= (m_1 + m_2 + m_3 + \ldots) R^2 = MR^2$$

- A solid disk of mass M, radius R, with axis through the center:

$I_{disk} = (1/2)\,MR^2$ (hard to show)



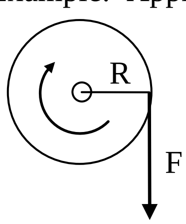
Moment of inertia I is a kind of "rotational mass".

Big I $\Rightarrow$ hard to get rotating (like Big M $\Rightarrow$ hard to get moving)

If I is big, need a big torque τ to produce angular acceleration according to

$$\tau_{net} = I \cdot \alpha \qquad (\text{ like } F_{net} = m\,a\ )$$

Example: Apply a force F to a pulley consisting of solid disk of radius R, mass M. α = ?



$$\tau = I\,\alpha$$

$$R\,F = \left(\tfrac{1}{2} MR^2\right)\alpha \quad \Rightarrow \quad \alpha = \frac{2F}{MR}$$

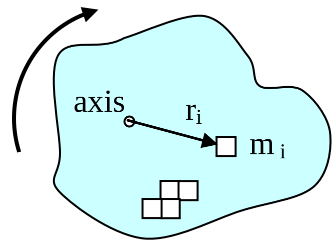
Rotational Kinetic Energy

How much KE in a rotating object? Answer: $\boxed{KE_{rot} = \tfrac{1}{2} I \omega^2}$ (like $KE_{trans} = \tfrac{1}{2} m v^2$)

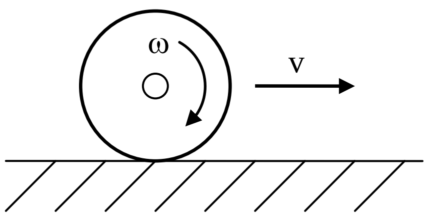
Proof: $KE_{tot} = \sum_i \left(\frac{1}{2} m_i v_i^2\right)$

$v = \omega r\,, \qquad v_i = \omega r_i$

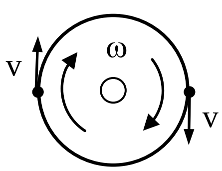
$$KE = \sum_i \left(\tfrac{1}{2} m_i \omega r_i^2\right) = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \tfrac{1}{2} I\,\omega^2$$



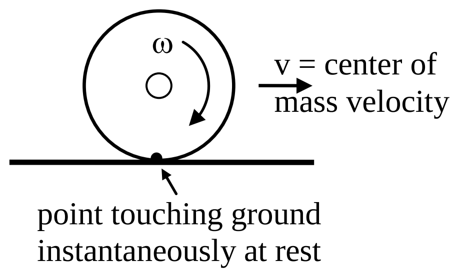
How much KE in a *rolling* wheel?



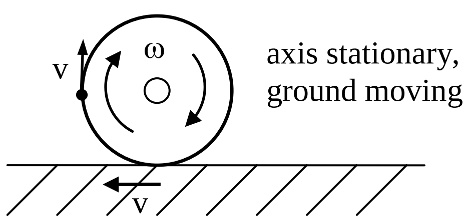
The formula v = r ω is true for a wheel spinning about a fixed axis or rolling on the ground.



axis stationary



To see why, look at situation from the bicyclist's point of view:



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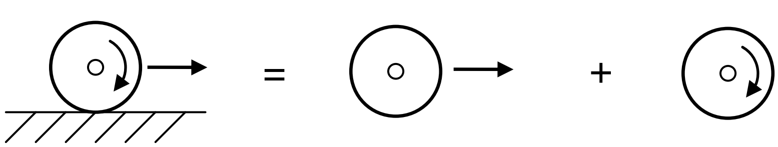
Rolling KE: Rolling wheel simultaneously translating and rotating:

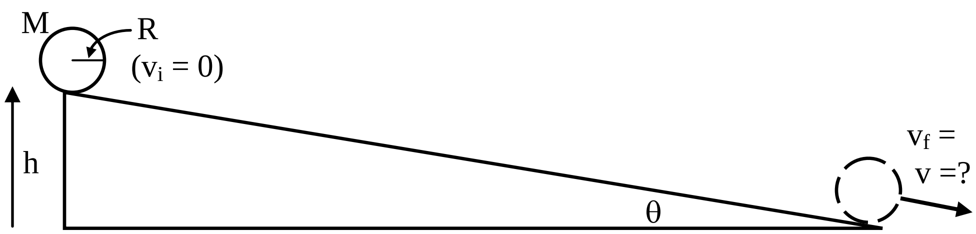


$KE_{tot} \quad = \quad KE_{trans} \quad + \quad KE_{rot}$

$$KE_{tot} = \tfrac{1}{2} M v^2 + \tfrac{1}{2} I\,\omega^2$$

Conservation of energy problems with rolling motion:

A sphere, a hoop, and a cylinder, each with mass M and radius R, all start from rest at the top of an inclined plane and roll down to the bottom. Which object reaches the bottom first?



Apply Conservation of Energy to determine v_{final}. Largest v_{final} will be the winner.

$$KE_i + PE_i = KE_f + PE_f$$

$$0 + Mgh = \underbrace{\tfrac{1}{2}Mv^2}_{\text{KEtrans}} + \underbrace{\tfrac{1}{2}I\omega^2}_{\text{KErot}} + 0$$

Value of moment of inertia I depends on the shape of the rolling thing:

$I_{disk} = (1/2)M R^2$, $I_{hoop} = M R^2$, $I_{sphere} = (2 / 5)M R^2$ (Computing the coefficient can be messy.)

Let's consider a disk, with $I = (1/2)MR^2$. For the disk, the rotational KE is

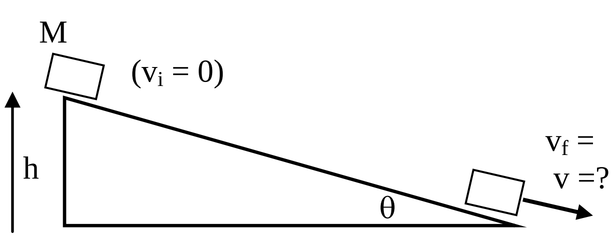
$$\tfrac{1}{2} I \omega^2 = \tfrac{1}{2}(\tfrac{1}{2} M R^2)\left(\frac{v}{R}\right)^2 = \tfrac{1}{4} M v^2 \qquad [\text{used } \omega = v/r]$$

$$\Rightarrow M g h = \tfrac{1}{2} M v^2 + \tfrac{1}{4} M v^2 = (\tfrac{1}{2}+\tfrac{1}{4}) M v^2 = \tfrac{3}{4} M v^2$$

$$g h = \tfrac{3}{4} v^2, \qquad v = \sqrt{\tfrac{4}{3} g h} \cong 1.16\sqrt{g h}$$

Notice that final speed does not depend on M or R.

Let's compare to final speed of a mass M, *sliding* down the ramp (no rolling, no friction).



$$M g h = \tfrac{1}{2} M v^2 \qquad \text{(M's cancel)}$$

$$\Rightarrow v = \sqrt{2 g h} \cong 1.4\sqrt{g h}$$

Sliding mass goes faster than rolling disk.

Why?

As the mass descends, PE is converted into KE. With a rolling object, $KE_{tot} = KE_{trans} + KE_{rot}$, so some of the PE is converted into KE_{rot} and less energy is left over for KE_{trans}. A smaller KE_{trans} means slower speed (since $KE_{trans} = (1/2)\, M\, v^2$). So rolling object goes slower than sliding object, because with rolling object some of the energy gets "tied up" in rotation, and less is available for translation.

Comparing rolling objects: $I_{hoop} > I_{disk} > I_{sphere} \Rightarrow$ Hoop has biggest $KE_{rot} = (1/2)\, I\, \omega^2$, $\Rightarrow$ hoop ends up with smallest $KE_{trans} \Rightarrow$ hoop rolls down slowest, sphere rolls down fastest.

Angular Momentum = "Spin"

Definition of *angular momentum* of a spinning object: $\boxed{L \equiv I\, \omega}$ (like $p = m\, v$)

If something has a big moment of inertia I and is spinning fast (big ω), then it has a big "spin", big angular momentum. Angular momentum is a very useful concept, because angular momentum is conserved.

Conservation of Angular Momentum: If a system is isolated from external torques, then its total angular momentum L is constant.

$$\tau_{ext} = 0 \Rightarrow L_{tot} = \text{constant} \qquad (\text{ like } F_{ext} = 0 \Rightarrow p_{tot} = \text{constant})$$

Here is a plausibility argument for conservation of angular momentum (proof is a bit too messy):

First, we argue that $\tau_{net} = \dfrac{\Delta L}{\Delta t}$ (like $F_{net} = \dfrac{\Delta p}{\Delta t}$) ,

$$\tau_{net} = I\,\alpha = I\,\frac{\Delta\omega}{\Delta t} = (\text{assuming I const})\ \frac{\Delta(I\omega)}{\Delta t} = \frac{\Delta L}{\Delta t}$$

(This turns out to be true even if $I \neq$ constant)

So now $\tau_{net} = \dfrac{\Delta L}{\Delta t} \Rightarrow$ if $\tau = 0$, then $\dfrac{\Delta L}{\Delta t} = 0 \Rightarrow L = \text{constant}$

It turns out that only 4 things are conserved:

- Energy
- Linear momentum p
- Angular momentum L
- Charge q

Conservation of Angular Momentum is very useful for analyzing the motion of spinning objects isolated from external torques — like a skater or a spinning star.

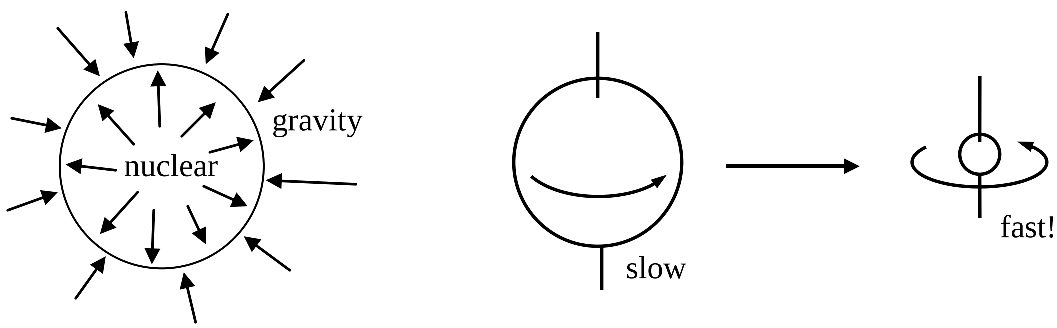
If $\tau_{ext} = 0$, then $L = I\,\omega$ = constant. If I decreases, ω must increase to keep L = constant.

Example: spinning skater.

$I_i\ \omega_i \quad = \quad I_f\ \omega_f$

(I big, ω small) (I small, ω big)

Example: rotation of collapsing star. A star shines by converting hydrogen (H) into helium (He) in a nuclear reaction. When the H is used up, the nuclear fire stops, and gravity causes the star to collapse inward.



As the star collapses (pulls its arms in), the star rotates faster and faster. Star radius can get **much** smaller: $R_i \approx 1$ million miles $\rightarrow$ $R_f \approx 30$ miles

$I_i\,\omega_i = I_f\,\omega_f$ (Sphere $I = \frac{2}{5} M R^2$)

$$\left(\tfrac{2}{5} M R_i^{\,2}\right)\omega_i = \left(\tfrac{2}{5} M R_f^{\,2}\right)\omega_f$$

$$R_i^{\,2}\,\omega_i = R_f^{\,2}\,\omega_f$$

$$\frac{R_i^{\,2}}{R_f^{\,2}} = \frac{\omega_f}{\omega_i} = \frac{T_i}{T_f} \qquad \left(\text{ using } \omega = 2\pi f = \frac{2\pi}{T}\right)$$

If $R_i >> R_f$, then $T_i >>>> T_f$.

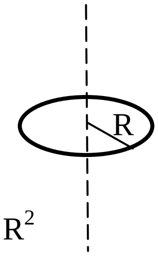
The sun rotates once every 27 days. "Neutron stars" with diameter of about 30 miles typically rotates 100 time per second.

Let's review the correspondence between translational and rotational motion

Translation	$\leftrightarrow$	**Rotation**
x	$\leftrightarrow$	θ
$v = \dfrac{\Delta x}{\Delta t}$	$\leftrightarrow$	$\omega = \dfrac{\Delta \theta}{\Delta t}$
$a = \dfrac{\Delta v}{\Delta t}$	$\leftrightarrow$	$\alpha = \dfrac{\Delta \omega}{\Delta t}$
F	$\leftrightarrow$	$\tau = r F_{\perp}$
M	$\leftrightarrow$	$I = \sum m r^2$
$F_{net} = M a$	$\leftrightarrow$	$\tau_{net} = I \alpha$
$KE_{trans} = (1/2) M v^2$	$\leftrightarrow$	$KE_{rot} = (1/2) I \omega^2$
$p = m v$	$\leftrightarrow$	$L = I \omega$
$F_{net} = \Delta p / \Delta t$	$\leftrightarrow$	$\tau_{net} = \Delta L / \Delta t$
If $F_{ext} = 0$, p_{tot} = constant	$\leftrightarrow$	If $\tau_{ext} = 0$, L_{tot} = constant

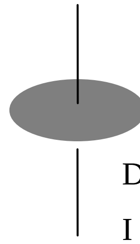
Appendix:

Moments of Inertia for some shapes:



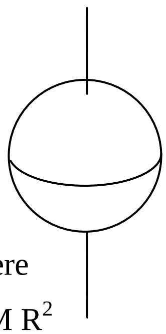
Hoop

$I = M R^2$



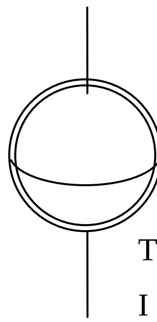
Disk

$I = (1/2) M R^2$



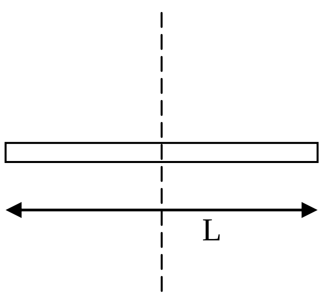
Solid sphere

$I = (2/5) M R^2$



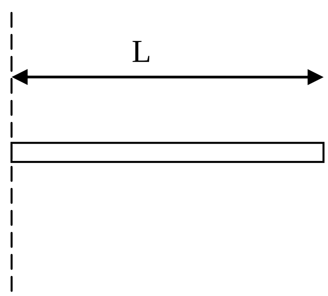
Thin spherical shell

$I = (2/3) M R^2$



Thin rod, axis thru center

$I = (1/12) M L^2$



Thin rod, axis thru end

$I = (1/3) M L^2$