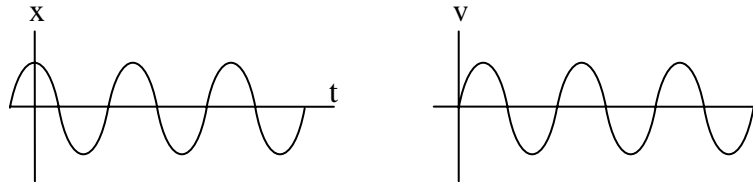


Simple Harmonic Motion

A pendulum, a mass on a spring, and many other kinds of oscillators exhibit a special kind of oscillatory motion called Simple Harmonic Motion (SHM).

SHM occurs whenever :

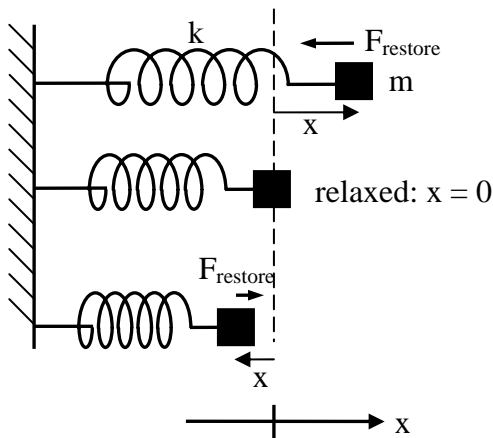
- there is a restoring force proportional to the displacement from equilibrium.
- the period T or frequency $f = 1 / T$ is independent of the amplitude of the motion.
- the position x , the velocity v , and the acceleration a are all sinusoidal (or harmonic) in time.



Any one of these three properties guarantees the other two. If one of these 3 things is true, then the oscillator is a Simple Harmonic Oscillator and all 3 things must be true.

Not every kind of oscillation is SHM. For instance, a perfectly elastic ball bouncing up and down on a floor: the ball's position (height) is oscillating up and down, but none of the 3 conditions above is satisfied, so this is not an example of SHM.

A mass on a spring is the simplest kind of Simple Harmonic Oscillator.



Hooke's Law: $\mathbf{F}_{\text{spring}} = -k \mathbf{x}$

(-) sign because direction of $\mathbf{F}_{\text{spring}}$ is opposite to the direction of displacement vector \mathbf{x}

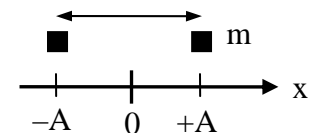
k = spring constant = stiffness,
units $[k] = \text{N} / \text{m}$

Big k = stiff spring

Recall $PE_{\text{elastic}} = (1/2) k x^2 =$ work done to compress or extend spring by distance x .

Definition: *amplitude* $A = |x_{\text{max}}| = |x_{\text{min}}|$.

Mass oscillates between extreme positions $x = +A$ and $x = -A$



SHM and Conservation of Energy:

If no friction, then total energy $E_{\text{tot}} = \text{KE} + \text{PE} = \text{constant}$ during oscillation. The value of E_{tot} depends on initial conditions – where the mass is and how fast it is moving initially. But once the mass is set in motion, E_{tot} stays constant (assuming no dissipation.)

At any position x , speed v is such that $\boxed{\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = E_{\text{tot}}}$

When $|x| = A$, then $v = 0$, and all the energy is PE: $\underbrace{\text{KE}}_0 + \underbrace{\text{PE}}_{(1/2)kA^2} = E_{\text{tot}}$

So total energy $E_{\text{tot}} = \frac{1}{2} k A^2$

When $x = 0$, $v = v_{\text{max}}$, and all the energy is KE: $\underbrace{\text{KE}}_{(1/2)m v_{\text{max}}^2} + \underbrace{\text{PE}}_0 = E_{\text{tot}}$

So, total energy $E_{\text{tot}} = \frac{1}{2} m v_{\text{max}}^2$.

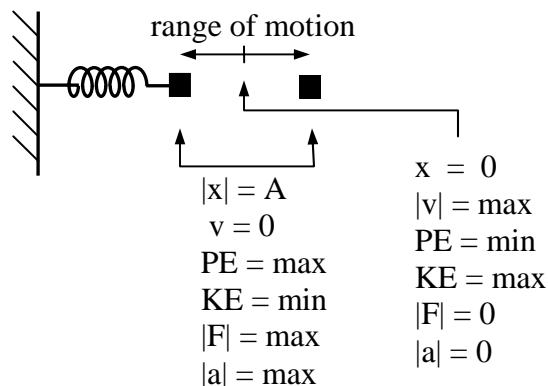
So, can relate v_{max} to amplitude A : $\text{PE}_{\text{max}} = \text{KE}_{\text{max}} = E_{\text{tot}} \Rightarrow \frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2 \Rightarrow$

$$v_{\text{max}} = \sqrt{\frac{k}{m}} A$$

Example Problem: A mass m on a spring with spring constant k is oscillating with amplitude A . Derive a general formula for the speed v of the mass when its position is x .

Answer: $v(x) = A \sqrt{\frac{k}{m}} \sqrt{1 - \left(\frac{x}{A}\right)^2}$

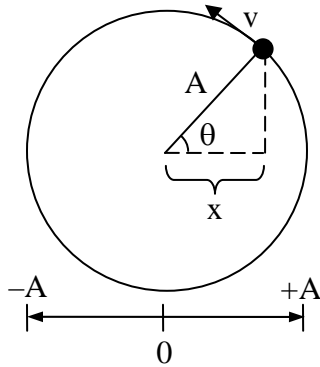
Understand these things:



SHM and circular motion

It turns out that there is an exact analogy between SHM and *circular motion*. Consider a particle moving with constant speed v around the rim of a circle of radius A .

The x -component of the position of the particle has *exactly* the same mathematical form as the motion of a mass on a spring executing SHM with amplitude A .



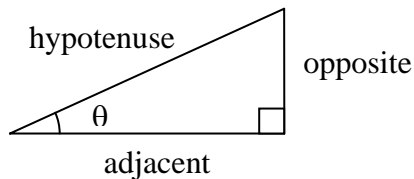
$$\text{angular velocity } \omega = \frac{\Delta\theta}{\Delta t} \Rightarrow$$

$$\theta = \omega t \quad \text{so}$$

$$x = A \cos \theta = A \cos \omega t$$

This same formula also describes the *sinusoidal* motion of a mass on a spring.

Let's review the sine and cosine functions and their relation to the *unit circle*. We often define the sine and cosine functions this way:

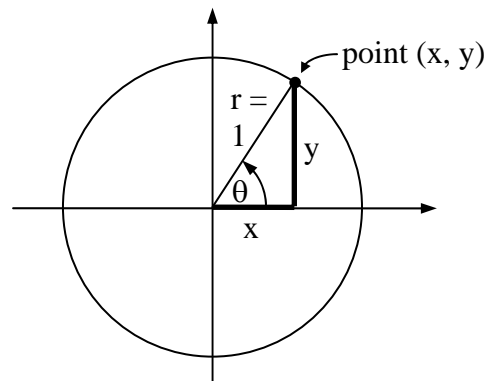


$$\cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

This way of defining sine and cosine is correct but incomplete. It is hard to see from this definition how to get the sine or cosine of an angle greater than 90° .

A more complete way of defining sine and cosine, a way that gives the value of the sine and cosine for *any* angle, is this: Draw a *unit circle* (a circle of radius $r = 1$) centered on the origin of the x - y axes as shown here:



Define sine and cosine as

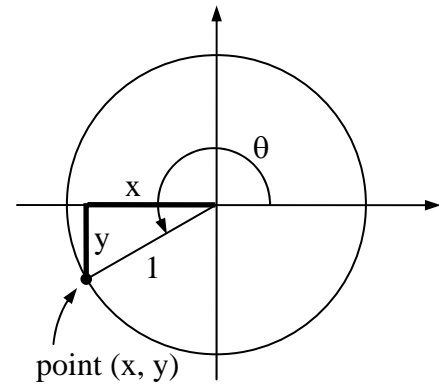
$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

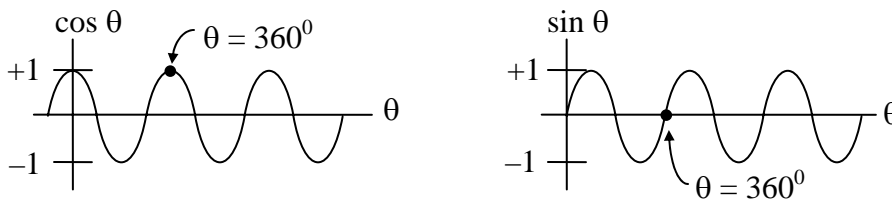
This way of defining sin and cos allows us to compute the sin or cos of *any* angle at all.

For instance, suppose the angle is $\theta = 210^\circ$. Then the diagram looks like this:

The point on the unit circle is in the third quadrant, where both x and y are negative. So both $\cos\theta = x$ and $\sin\theta = y$ are negative



For any angle θ , even angles bigger than 360° (more than once around the circle), we can always compute \sin and \cos . When we plot \sin and \cos vs angle θ , we get functions that oscillate between $+1$ and -1 like so:



We will almost always measure angle θ in radians. Once around the circle is 2π radians, so sine and cosine functions are periodic and repeat every time θ increases by 2π rad. The sine and cosine functions have exactly the same shape, except that \sin is shifted to the right compared to \cos . Both these functions are called *sinusoidal* functions.

Now back to simple harmonic motion. Instead of a circle of radius 1, we have a circle of radius A (where A is the amplitude of the Simple Harmonic Motion).

We aim to show that, for a simple harmonic oscillator consisting of a mass m on spring with constant k , if the period is T , then the position as a function of time t is given by:

$$x = A \cos(\omega t) \quad \text{where } \omega \equiv \frac{2\pi}{T} \quad \text{and, furthermore, } \omega \text{ is related to the } k \text{ and } m \text{ by}$$

$$\omega = \sqrt{\frac{k}{m}}$$

Let's first try to make sense of this: big ω means small T which means rapid oscillations. According to our formula $\omega = \sqrt{k/m}$, we get a big ω when k is big and m is small. This makes sense: a big k (stiff spring) and a small mass m will indeed produce very rapid oscillations and a big ω .

Ok, so where does this formula $\omega = \sqrt{k/m}$ come from and what's the connection with circular motion?

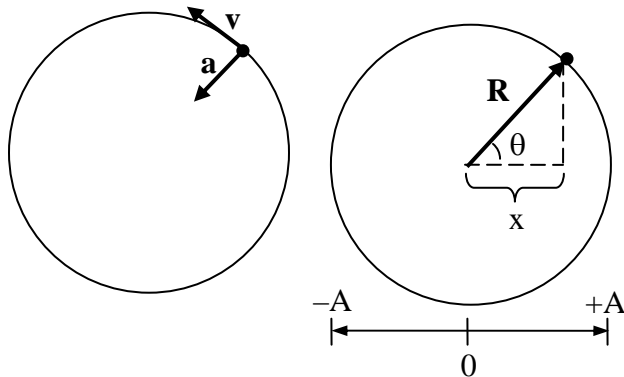
Newton's 2nd law (in 1D) is $F_{\text{net}} = m a$. Applying this to a mass on a spring ($F_{\text{net}} = -kx$) we get

$$-k x = m a \quad \text{or} \quad a = -\frac{k}{m} x \quad (1)$$

This equation says: (acceleration a) = – (positive constant) \times (position x). This is called an *equation of motion*. Anytime we have this equation of motion, then we have SHM since this equation is equivalent to the condition $|F_{\text{restore}}| \propto |\text{displacement from equilibrium}|$.

We can show that the x-component (or the y-component) of a particle moving with constant speed v around a circle obeys the *same* equation (1) [$a = -(k/m)x$]. Recall that for circular motion with angular speed ω , the acceleration of a the particle is toward the center and has magnitude

$$|\vec{a}| = \frac{v^2}{R}, \quad \text{but } v = \omega R, \quad \text{so we can rewrite this as } |\vec{a}| = \frac{(\omega R)^2}{R} = \omega^2 R$$



Notice that the acceleration vector \vec{a} is always in the direction opposite the displacement vector \vec{R} . In vector language, $\vec{a} = -\omega^2 \vec{R}$. The x-component of this vector equation is: $a_x = -\omega^2 R_x$. If we write $R_x = x$, then we have

$$a_x = -\omega^2 x = -\omega^2 A \cos \theta = -\omega^2 A \cos \omega t. \quad (2)$$

This equation (2) is has the same form as equation (1):

(acceleration) = – (positive constant) \times (position).

Comparing the positive constants in the two equations (1) and (2), we see the equations

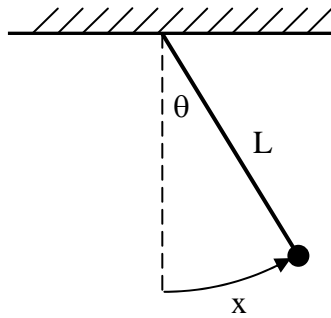
are identical if we set $\omega^2 = \frac{k}{m}$.

$$\text{Notice } \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}} \Rightarrow T = 2\pi \sqrt{\frac{m}{k}}$$

Notice that period T is *independent* of the amplitude A ; it depends only on the mass m and the spring constant k .

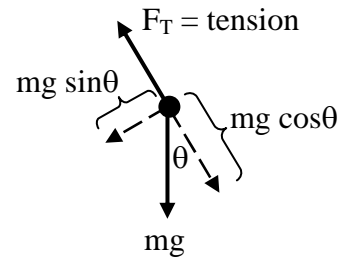
Pendulum Motion

A simple pendulum consists of a small mass m suspended at the end of a massless string of length L . A pendulum executes SHM, if the amplitude is not too large.



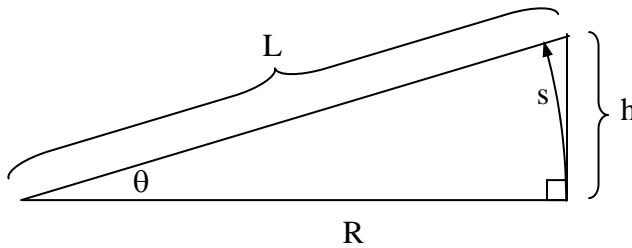
$$\theta = x / L \text{ (rads)}$$

Forces on mass :



$$\text{restoring force} = -mg \sin \theta \cong -mg \theta = -mg \frac{x}{L}$$

$$\text{Claim: } \sin \theta \cong \theta \text{ (rads) when } \theta \text{ is small. } \sin \theta = \frac{h}{L}$$



$$\theta = \frac{s}{R}$$

If θ small, then $h \approx s$, and $L \approx R$,
so $\sin \theta \approx \theta$.

$$\text{Try it on your calculator: } \theta = 5^\circ = 0.087266.. \text{ rad}$$

$$\sin \theta = 0.087156..$$

$$F_{\text{restore}} = -\left(\frac{mg}{L}\right)x \text{ is exactly like Hooke's Law } F_{\text{restore}} = -kx, \text{ except we have}$$

replace the constant k with another constant (mg/L) . The math is exactly the same as with a mass on a spring; all results are the same, except we replace k with (mg/L) .

$$T_{\text{spring}} = 2\pi \sqrt{\frac{m}{k}} \Rightarrow T_{\text{pend}} = 2\pi \sqrt{\frac{m}{(mg/L)}} = 2\pi \sqrt{\frac{L}{g}}$$

Notice that the period is independent of the amplitude; the period depends only on length L and acceleration of gravity. (But this is true only if θ is not too large.)