- We call $\Psi(x,t)$ the probability (density) amplitude. The wavefunction contains the whole information about the particle.
  But, $\Psi(x,t)$ itself has <u>no empirical</u> significance only $|\Psi(x,t)|^2$ as being the probability density has it.

But, which equation determines $\Psi(x,t)$?

Historical note:
As a Professor of Physics at Zürich University Erwin Schrödinger gave a colloquium talk on how de Broglie associated a wave with a particle. When he had finished his collegue Peter Debye remarked that he thought that this theory is childish because "to deal properly with waves, one had to have a wave equation" (see Felix Bloch, Physics Today, December 1976, p. 23)

Just a few weeks later Erwin Schrödinger gave another talk in the colloquium which he started by saying: "My colleague Debye suggested that one should have a wave equation; well, I have found one!"

4.5 Schrödinger equation

The wave equation to describe the state of a particle cannot be derived from fundamental equations. But, we can (Schrödinger could) make an educated guess. Requirements for equation:

- Solution of wave equation must have wavelength that satisfy the de Broglie relation
$$\lambda = h/p = \frac{h}{\sqrt{2m(KE)}}$$

- Solution must satisfy the equations for total, kinetic and potential energy
$$TE = KE + PE = \frac{p^2}{2m} + V(x)$$

- Solution must satisfy the superposition principle. If Ψ_1 is a solution and Ψ_2 is a solution of the wave equation, then $\Psi = \Psi_1 + \Psi_2$ must be a solution as well.

  $\Rightarrow$ Equation must be a linear homogeneous differential equation.

The time-dependent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

while the time-independent Schrödinger equation is

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x)$$

For a potential $V(x)$ that does depend on x only (but not on time t), the solutions for the time-dependent and the time-independent equation are related as:

$$\Psi(x,t) = \psi(x)\ \phi(t)$$

i.e. can be written as product

An example for a solution of the time-independent Schrödinger equation is

$$\psi(x) = \psi_0 \sin(kx) \qquad \text{with} \quad k = \frac{2\pi}{\lambda}$$

Proof:

LHS: $$\frac{d^2}{dx^2}\psi(x) = \psi_0 \left(-\frac{(2\pi)^2}{\lambda}\right) \sin(kx) = -\frac{(2\pi)^2}{\lambda^2}\psi_0 \sin(kx)$$

RHS: $$-\frac{2m}{\hbar^2}\underbrace{\left[E - V(x)\right]}_{= KE \text{ (see one of requirements)}}\psi_0 \sin(kx)$$

$$= -\frac{2m(KE)}{\hbar^2}\psi_0 \sin(kx)$$

$$= -\frac{h^2}{\hbar^2\lambda^2}\psi_0 \sin(kx) = -\frac{(2\pi)^2}{\lambda^2}\psi_0 \sin(kx) \checkmark$$

$2m(KE) = \frac{h^2}{\lambda^2}$