For the time-dependent Schrödinger equation we first consider the case of $V(x) = 0$, that is a free particle (no potential), i.e.

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x,t) = i\hbar \, \Psi(x,t)$$

A solution of the time-dependent Schrödinger equation of a free particle is given by

$$\Psi(x,t) = e^{i(kx - \omega t)}$$

Proof:

LHS: $$-\frac{\hbar^2}{2m} (ik)^2 e^{i(kx-\omega t)} = \frac{\hbar^2 k^2}{2m} e^{i(kx-\omega t)}$$

$$\underset{k = p/\hbar}{=} \frac{p^2}{2m} e^{i(kx-\omega t)} = \underbrace{E}_{= KE} e^{i(kx-\omega t)}$$

RHS: $$i\hbar(-i\omega) e^{i(kx-\omega t)} = \hbar\omega \, e^{i(kx-\omega t)}$$

$$\underset{\omega = 2\pi f}{=} \frac{h}{2\pi} 2\pi f \, e^{i(kx-\omega t)}$$

$$\underset{E = hf}{=} E \, e^{i(kx-\omega t)}$$

Remarks:

- The wavefunction $\Psi(x,t)$ is complex and has therefore no physical significance
  $\rightarrow$ no observable

- Square of the wavefunction is an observable since we interpret it as a probability density.
  The correct way to take the square is

$$\rightarrow \quad |\Psi|^2 = \Psi\,\Psi^*$$

complex conjugate.

since this is always a real number

while $\Psi^2 = \Psi\,\Psi$ is not good, since it is, in general, a complex number.

Excursus : Complex numbers

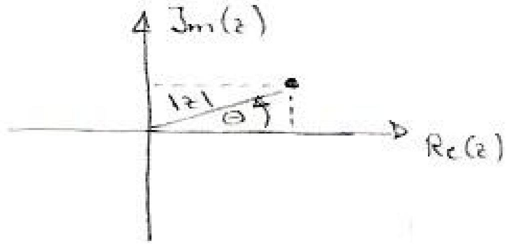
Representation of complex numbers.

A complex number is given by

$$z = x + iy = Re(z) + i\, Im(z)$$

$$\text{where } i^2 = -1 \quad \text{and} \quad i = \sqrt{-1}$$

It can be represented in the complex plane as:



Alternative representation is given by magnitude $|z|$ and the angle θ in the complex plane

Relations:

$$x = |z| \cos\theta \qquad |z| = \sqrt{x^2 + y^2}$$

$$y = |z| \sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$\Rightarrow z = |z| \left(\cos\theta + i \sin\theta \right)$$

Important formulas:

$$e^{i\theta} = \cos\theta + i \sin\theta \quad \text{: Euler formula}$$

$$z^* = x - iy \quad \text{: Complex conjugate.}$$

Multiplication:

Examples: $z^2 = z\,z = (x+iy)(x+iy)$

$$= x^2 + y^2 + 2iyx \quad : \text{complex}$$

$$|z|^2 = z\,z^* = (x+iy)(x-iy)$$

$$= x^2 + y^2 = |z|^2 \quad : \text{real}$$

Derivatives:

Remember: $\frac{d}{dx} e^{ax} = a\, e^{ax}$

$$\Rightarrow \frac{d}{dx} e^{ikx} = ik\, e^{ikx}$$

$$\Rightarrow \frac{\partial}{\partial x} e^{i(kx-\omega t)} = ik\, e^{i(kx-\omega t)}$$

$$\frac{\partial}{\partial t} e^{i(kx-\omega t)} = -i\omega\, e^{i(kx-\omega t)}$$