4.6 Plane waves, wave packets and uncertainty relation

As we have seen in the last section a solution of the time-dependent Schrödinger equation for a free particle $(V = 0)$ is given by

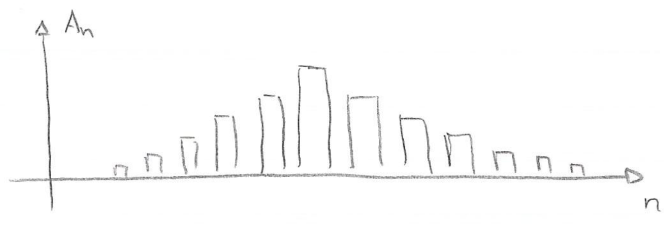
$$\Psi(x,t) = \exp\left[i(kx - \omega t) \right]$$

There is no restriction on the value of k. Therefore for any value of k the above wave function is a solution and, hence, also any superposition

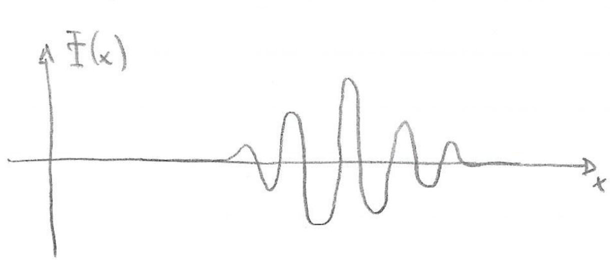
$$\Psi(x,t) = \sum_n A_n \exp\left[i(k_n x - \omega_n t) \right]$$

is a solution. And, this means that it describes a possible state of a particle.

If we choose the coefficients (approximately) as follows

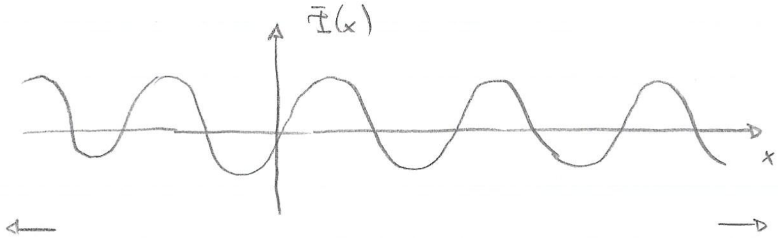


then the superposition looks like



wave packet

as compared to $\Psi(x,t) = \exp\left[i(kx - \omega t)\right]$



plane wave

extends to $-\infty$ extends to ∞

The superposition is called wave packet in which the particle is localized in a small interval, while the individual solution is called plane wave.

Comparing both solutions (states of particle)

Plane wave: just one momentum → precise knowledge
no uncertainty
no info about position → large uncertainty

Wave packet: localized in certain interval of x
→ less uncertainty in position
superposition of many waves (momenta)
→ greater uncertainty in momentum

Thus, we see that an
increase in localization of particle comes with a
decrease in knowledge of momentum of particle

This is a general principle in quantum mechanics which is summarized in the Heisenberg uncertainty relation

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

or in words: The product of uncertainty in position (Δx) and uncertainty in momentum (Δp) cannot be smaller than a certain limit ($\hbar/2$).

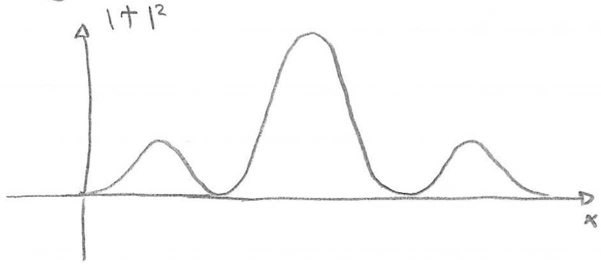
Physically, position and momentum of a particle cannot be determined with complete precision at the same time.

The uncertainty relation tells us about the precision of measurements of position and momentum simultaneously. It does not restrict the precision of an individual measurement of either position or momentum.

However, the underlying principle of localization and delocalization of the particle still has implications on position and momentum measurements. Let's see what happens by looking at an example:

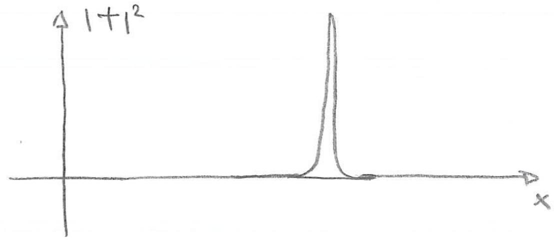
Assume the state of the particle is given by the following probability density $|\Psi(x,t)|^2$

Before measurement:



In a position measurement the particle can be found at any localization, where the probability density is not equal to zero. Once the position of the particle has been measured, the probability density must be concentrated around the measured localization

After position measurement:



Based on our previous analysis; we have

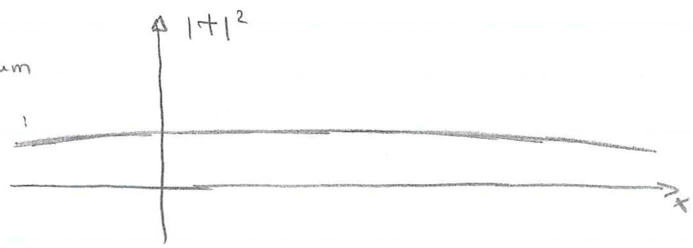
→ small uncertainty in position

→ large uncertainty in momentum

described by a strongly localized wave packet

Next, after some time we measure the momentum of the particle. Once the momentum is measured, the probability density must look like

After momentum measurement:



Now, we have a small uncertainty in momentum but a large uncertainty in position of particle, described by a plane wave.