

Spin of atoms

In general, projection of magnetic moment vector onto a given axis can have more than two possible values. This occurs since in general the magnetic moment of atoms is due to both orbital angular momentum and spin (angular momentum).

In order to further study the measurement of complementary variables (here, projections of spin), we will consider two-state systems only, i.e. as in the original Stern-Gerlach experiment.

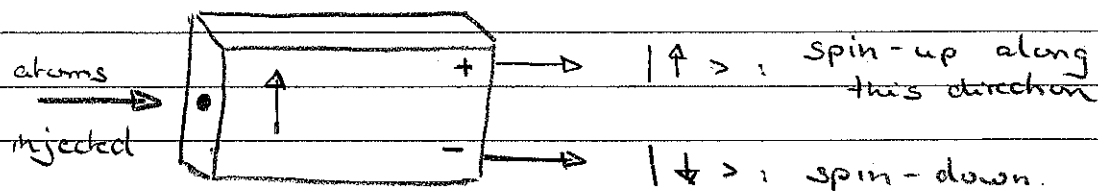
Notation: An atom is either measured as

spin-up : $|\uparrow\rangle$ or $|+\frac{1}{2}\rangle$ or

spin-down : $|\downarrow\rangle$ or $|-\frac{1}{2}\rangle$

along a given axis.

For the measurements we will consider a simplified Stern-Gerlach analyzer



For the injection of atoms we can consider two options

- Ensemble of atoms with randomly oriented spin (magnetic moment)

⇒ Atoms will exit with equal probability of $\frac{1}{2}$ the plus and minus channels

- Ensemble of atoms in a "prepared spin (magnetic moment)" state

Examples:

- (i) Atoms are prepared in spin-up state along the z-direction and are analyzed with a spin-up (along z) Stern-Gerlach analyzer

Result: Probability to exit + channel: 100%

Probability to exit - channel: 0%

since atoms are prepared in the state along the axis that is analyzed.

- (ii) Atoms are prepared in spin-up (spin-down) state along x-direction (or y-direction) and are analyzed with a spin-up or spin-down (along z) analyzer

Result: Probability to exit + channel: 50%

Probability to exit - channel: 50%

since atoms are prepared in a state complementary to the one analyzed.

(iii) Atoms are prepared in a spin-up state along a direction at an angle of θ to the axis analyzed.

Result: Probability to exit + channel: $\cos^2\left(\frac{\theta}{2}\right)$

Probability to exit - channel: $\sin^2\left(\frac{\theta}{2}\right)$

Note: $\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) = 1$

Remarks about repeated spin measurements:

• Total probability for atom to pass successive Stern-Gerlach analyzers along a certain path is given by the product of the individual probabilities.

• Expectation value (Average value) of measurement is

$$\langle x \rangle = \sum_{i=1}^2 P(x_i) x_i$$

probability

for individual result.

individual possible results

5. One-dimensional problems

In order to analyze quantum mechanical problems we need to find the solution of the time-dependent Schrödinger equation for the specific problem:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x,t) \Psi(x,t) = i\hbar \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

The solution $\Psi(x,t)$ contains all information about the particle (system). Itself it does however not have a physical meaning. We can retrieve information e.g. via

$|\Psi(x,t)|^2$: probability density

$\langle \Psi(x,t) \rangle$: expectation (average) value.

We will consider problems with $V(x,t) = V(x)$, i.e. the potential is independent of time. This is not a big restriction, indeed for most quantum mechanical problems the potential is time-independent. For example, the Coulomb potential for charged particles inside an atom or a molecule is time-independent.

In this case ($V(x,t) = V(x)$) it can be shown that the solution is always written as:

$$\Psi(x,t) = \psi(x) \phi(t)$$

i.e. the solution separates in a product in which one function ($\psi(x)$) depends on position only while the other ($\phi(t)$) depends on time only. We call this also separation of variables.

Furthermore: $\phi(t) = \exp\left(-iEt/\hbar\right)$

where E is the total energy of the particle (system).

On the other hand, $\psi(x)$ is solution of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x)$$

and it is our task to solve this differential equation for a given problem. Erwin Schrödinger was able to do so for the hydrogen atom and got the energy values, that are in agreement with the Balmer series.

5.1 Solving the (time-independent) Schrödinger equation

We will now study the general concept to find a solution of the Schrödinger equation for a given quantum mechanical problem. We will separate this general strategy into several steps:

Step 1: Analyze the system and determine a potential energy function $V(x)$.

This is an important first step since $V(x)$ is needed to establish the Schrödinger equation. In order to find a solution we often need to make simplifying assumptions, This is needed