Step 5 : Full time-dependent solution

Putting everything together we get :

$$\Psi(x,t) = \psi(x) \exp\left(-i \frac{Et}{\hbar}\right) \qquad \text{for } 0 \leq x \leq L$$

$$= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \exp\left(-i \frac{E_n t}{\hbar}\right)$$

$$\text{where } \frac{2m E_n}{\hbar^2} = k_n^2 = \frac{n^2 \pi^2}{L^2}$$

$$\Rightarrow E_n = n^2 \frac{\pi^2 \hbar^2}{2mL^2}$$

This is the mathematical solution of the Schrödinger equation for the infinite square well.

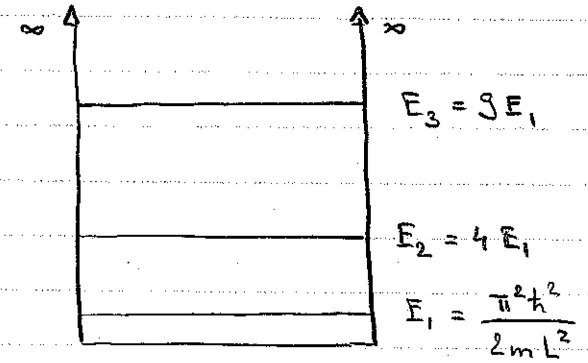
Now, we can interpret the physics of this solution :

- There are only certain stationary states with discrete total energies possible, namely

$$E_n = n^2 \frac{\pi^2 \hbar^3}{2mL^2} \qquad \text{with } n = 1, 2, 3, 4, \ldots$$

This is in contrast to a classical counterpart in which the particle could have any energies.

Sketch of total energies (in potential well):

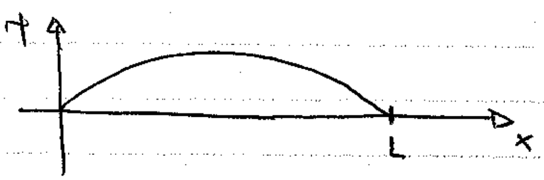


Note that the potential energy for each of these discrete energy levels is $V=0$ (we have chosen it like this). Therefore, the total energy is always equal to the kinetic energy, as long as $V=0$ is chosen.

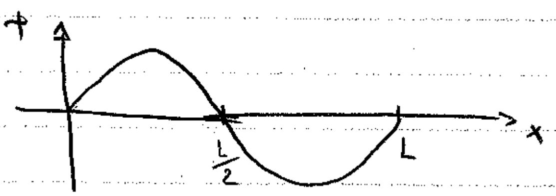
- How do the (spatial parts) of the wave functions look like?

$$\psi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi}{L}x\right) \qquad \text{with } n = 1, 2, 3, \ldots$$

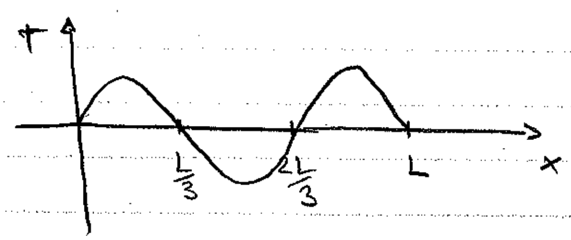
$\psi_1(x)$:



$\psi_2(x)$:



$\psi_3(x)$:



- Note that the total energy in the ground state ($n = 1$) cannot be zero, since otherwise the Heisenberg uncertainty relation would not be fulfilled.

We can estimate for the ground state

$$E_1 = \frac{p_1^2}{2m} \Rightarrow p_1 = \pm\sqrt{2mE_1}$$

$$= \pm\sqrt{\frac{\pi^2\hbar^2}{L^2}} = \pm\frac{\pi\hbar}{L}$$

$$\Rightarrow \Delta p \simeq \frac{\pi\hbar}{L} \quad \text{(this is an estimate !)}$$

and $\Delta x \simeq L/2$ (this is an estimate as well !)

$$\Rightarrow \Delta x \, \Delta p_1 \simeq \frac{L}{2}\frac{\pi\hbar}{L} = \frac{\pi\hbar}{2} \geq \frac{\hbar}{2} \quad \checkmark \quad \text{Good !}$$

In case of a total energy of zero the momentum would be zero as well and hence $\Delta p = 0$

This would mean $\Delta x \, \Delta p = 0$ which would violate the uncertainty relation !