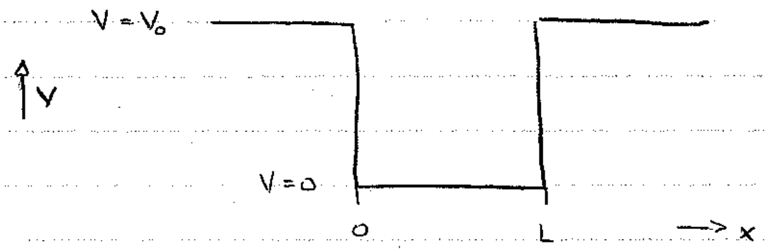
5.2 Finite square well

We will now extend our studies for a finite square well:



In the region $0 \le x \le L$ the solution is given (as before) by

$$\psi(x) = A \sin(kx) + B \cos(kx) \qquad \text{with} \quad k = \sqrt{\frac{m(E-0)}{\hbar}}$$

Outside of this region we consider two different cases
(note will not attempt to find the exact solution but we will discusss it qualitatively).

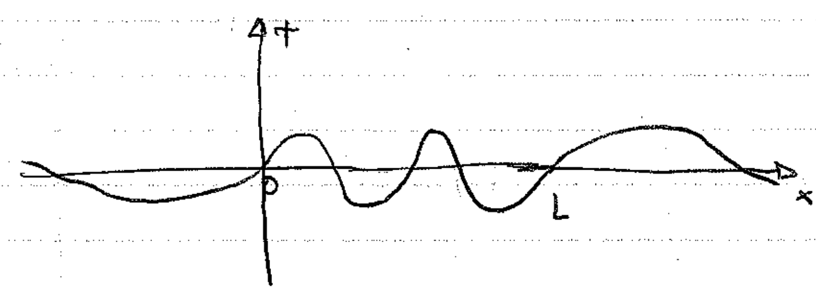
$E > V_0$: In this case the solution outside is also given by

$$\psi(x) = A_0 \sin(k_0 x) + B_0 \cos(k_0 x) \qquad \text{with} \quad k_0 = \sqrt{\frac{2m(E-V_0)}{\hbar}}$$

Thus, inside as well as outside we have oscillatory solutions, however with different wave numbers

$$k_{\text{inside}} < k_{\text{outside}} \Rightarrow \lambda_{\text{inside}} > \lambda_{\text{outside}}$$

Qualitative sketch:



$E < V_0$ (but $E > 0$): Now the Schrödinger equation in the region outside $0 \leq x \leq L$ is given by

$$\frac{d^2\psi}{dx^2} = +\beta^2 \psi(x) \quad \text{with} \quad \beta = \sqrt{\frac{2m\,(V_0 - E)}{\hbar}}$$

positive

Solutions are $\psi_1(x) = A_0 e^{\beta x}$ and $\psi_2 = B_0 e^{-\beta x}$

$\Rightarrow$ general solution: $\psi(x) = A_0 e^{\beta x} + B_0 e^{-\beta x}$ for $x < 0$

$\psi(x) = A_0 e^{\beta(x-L)} + B_0 e^{-\beta(x-L)}$ for $x > L$

Note that in the region

- $x < 0$: $e^{-\beta x}$ increases towards ∞ for $x \to -\infty$ and this is physically impossible

  $\Rightarrow \psi(x) = A_0 e^{\beta x}$ for $x < 0$

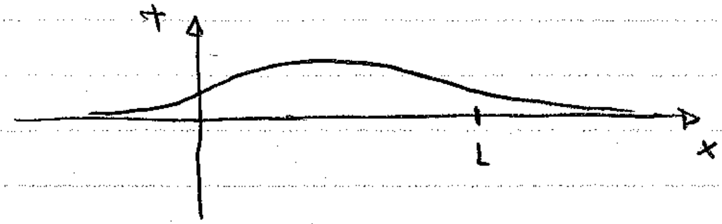
- $x > L$: $e^{\beta(x-L)}$ increases towards ∞ for $x \to \infty$ and this is physically impossible

  $\Rightarrow \psi(x) = B_0 e^{-\beta x}$ for $x > L$

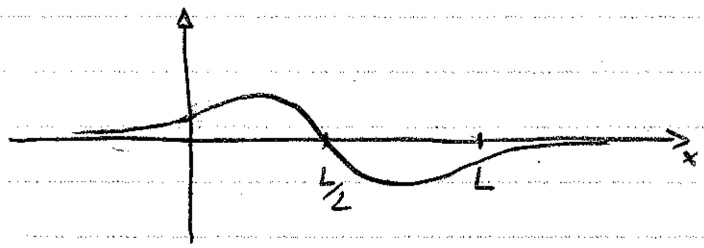
Thus, inside the well we have an oscillatory solution (as before for the infinite square well), while outside it is an exponentially decaying solution

Sketch of (stationary) wave functions for $E < V_0$:

$\psi_1(x)$:



$\psi_2(x)$:



<u>Notes</u>:

- For $E > V_0$ <u>all</u> energies are allowed.
  For $E < V_0$ only <u>discrete</u> energies are allowed.

- Quantum mechanically, the particle can <u>penetrate</u> into the classically forbidden regions $x < 0$ and $x > L$.

  One <u>defines</u> a characteristic distance = <u>penetration depth</u> as:

$$\frac{\psi(x = -1/\beta)}{\psi(x=0)} = \frac{\psi(x=0)\, e^{+\beta(-1/\beta)}}{\psi(x=0)} = e^{-1}$$

Thus, over the distance $\Delta x = 1/\beta$ the wavefunction <u>decays</u> by $e^{-1} = 1/e$

Note: $\Delta x = \frac{1}{\beta} = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$