1.2 Electromagnetic waves

Light is an electromagnetic wave, i.e. a coupled oscillating electric and magnetic field. To study this further, let's review:

An electric field exerts a force on a charge

A magnetic field exerts a force on a moving charge

$$\vec{F}_{\text{on } q} = q \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

properties of particle

Thus, Any electric and magnetic field interacts with matter by exerting a force on charged particles (on electrons and protons in atoms, molecules, solids, ...)

How are electric and magnetic fields generated?

Static electric field : by stationary charges

Static magnetic field : by charges moving at constant velocity (constant current through wire)

Changing electric fields / magnetic fields :

by accelerating charges

This is captured by the Maxwell equations

in differential form

$$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \qquad\qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

in integral form

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0} \qquad\qquad \oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{L} = -\frac{d\phi_B}{dt} \qquad\qquad \oint \vec{B} \cdot d\vec{L} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\phi_E}{dt}$$

From the Maxwell equations one can derive the so-called wave equation

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \left(\text{with } \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^3}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

which we will consider in 1D

$$\frac{\partial^2 \vec{E}}{\partial x^2} \pm \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

(analogous equation for magnetic field exist)

Light is an electromagnetic wave, that is a coupled oscillating electric and magnetic field. The oscillation of the fields is often described by a sinusoidal function.

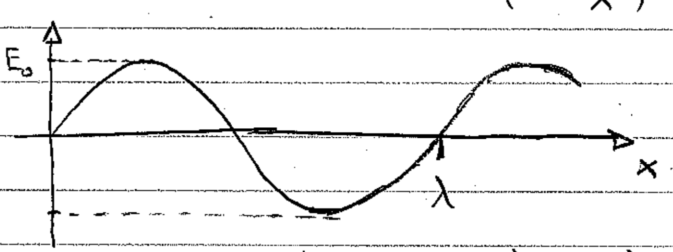
For electric field: $\vec{E}(x,t) = \underbrace{\vec{E}_0}_{\text{amplitude of field}} \sin(ax + bt)$

This expression is indeed a solution of the 1D wave equation, as we will see later, if

$$\vec{E}(x,t) = \vec{E}_0 \sin\left(2\pi \frac{x}{\lambda} - 2\pi \frac{t}{T}\right)$$

Before we check this, let's first understand this functional form.

at $t=0$: $\vec{E}(x, t=0) = \vec{E}_0 \sin\left(2\pi \frac{x}{\lambda}\right)$



$\vec{E}(x+\lambda, t=0) = \vec{E}(x, t=0)$ for all x

Thus, there is a periodicity in x with period λ, which we call wavelength

at $x=0$: $\vec{E}(x=0, t) = E_0 \sin\left(2\pi \frac{t}{T}\right)$

$\Rightarrow$ periodic in t with period T

$\vec{E}(x=0, t+T) = \vec{E}(x=0, t)$

Let's consider the full functional form

$$E(x,t) = E_0 \sin\left(2\pi \frac{x}{\lambda} - 2\pi \frac{t}{T}\right)$$

$$= E_0 \sin\left(2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right)$$

Introduce $v = \frac{\lambda}{T} > 0$; units of velocity

$$\Rightarrow \frac{t}{T} = \frac{vt}{\lambda}$$

$$= E_0 \sin\left(2\pi \frac{x - vt}{\lambda}\right) = E_0 \sin\left(2\pi \frac{x - \Delta x}{\lambda}\right)$$

matches description of wave at $t=0$ $E(x - \Delta x, t=0)$

Interpretation:

E-field at position x at time t ; first line

= E-field at position $x - \Delta x$ at time $t = 0$

$\Rightarrow$ E-field (Wave) moves from $t=0$ to t from $x - \Delta x = x - vt$ to x

Since $x - \underbrace{vt}_{\text{positive}} < x$ the E-field (the wave) moves in positive x-direction with velocity (speed) $v = \frac{\lambda}{T}$

[Try to figure how the function changes for a wave moving in negative x-direction]

One introduces now (to simplify)

wave number $k = 2\pi/\lambda$

angular frequency $\omega = 2\pi/T$

$$\Rightarrow \quad E(x,t) = E_0 \sin(kx - \omega t)$$

Now we will convince ourselves that such electromagnetic waves exist. Any (real) electric field has to fulfill the Maxwell equation or here the wave equation (which can be derived from the Maxwell equations)

$$\frac{\partial^2 E(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E(x,t)}{\partial t^2}$$

We will not work through the full proof that

$$E(x,t) = E_0 \sin(kx - \omega t)$$

indeed fulfills this equation, but we will motivate that it is the case. To this end, we will understand the left hand side of the equation. Let's start with

$$\frac{d}{dx} \sin(kx) = \cos(kx) \frac{d}{dx}(kx) = k \cos(x)$$

Next we consider the sin-function depending on x <u>and</u> t

$$\frac{\partial}{\partial x} \sin(kx - \omega t) = \cos(kx - \omega t) \frac{\partial}{\partial x}(kx - \omega t)$$

$$= k \cos(kx - \omega t)$$

partial derivative
since function depends
on x <u>and</u> t

And then we take the second derivative

$$\frac{\partial^2}{\partial x^2} \sin(kx - \omega t) = \frac{\partial}{\partial x} \Big(\underbrace{\frac{\partial}{\partial x} \sin(kx - \omega t)}_{= k \cos(kx - \omega t)} \Big)$$

$$= \frac{\partial}{\partial x} \big(k \cos(kx - \omega t) \big)$$

$$= -k^2 \sin(kx - \omega t)$$

Do you "see" that we get a similar expression on the RHS of the wave equation for the 2nd derivative in time.

→ <u>Looks</u> <u>promising</u>! Seems that such a sin-function fulfills the wave equation → Work it out!

Analogous you can show that $E(x,t) = E_0 \cos(kx - \omega t)$ is also a solution of the wave equation.