3220 Final Exam Cribsheet – DRAFT VERSION

Spin ½: $|\pm\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}$, $|\pm\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle \pm i|-\rangle) \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$

$S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, S_n \doteq \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta\, e^{i\phi} & -\cos\theta\end{pmatrix}$,

$S^2 \doteq \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Eigenvectors of S_n are $|+\rangle_n \doteq \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{pmatrix}$, $|-\rangle_n \doteq \begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{i\phi}\end{pmatrix}$.

Spin 1 systems:

$|1\rangle_x = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$, $|1\rangle_y = \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$

$|0\rangle_x = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$, $|0\rangle_y = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$,

$|-1\rangle_x = \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$, $|-1\rangle_y = \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$

$\hat{S}_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$, $\hat{S}_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$, $\hat{S}_z \doteq \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$

Postulate #4: If you make a quantum measurement on state $|\psi\rangle$ of an operator A the probability of getting one of its eigenvalues a_n as your outcome is $|\langle a_n|\psi\rangle|^2$

Postulate #5: If you measure A on state $|\psi\rangle$ and get one of its eigenvalues a_n, then you "collapse" into state: $|\psi_{new}\rangle = \frac{P_{a_n}|\psi\rangle}{\sqrt{\langle\psi|P_{a_n}|\psi\rangle}}$, where P_{a_n} (the "projection operator into state a_n") is defined as: $P_{a_n} = |a_n\rangle\langle a_n|$

Postulate #6: $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$ (The time-dependent Schrödinger Eqn)

Solution if H is time independent: $|\psi(t)\rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}}|E_n\rangle$

Completeness (or "closure") says$\sum_n P_{a_n} = \mathbf{1}$: , or in continuous basis $\int |a\rangle\langle a|\, da = \mathbf{1}$

Quantum expectation value $\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_n a_n \,\mathrm{Prob}(a_n)$

Quantum uncertainty: $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

QM uncertainty principle: $\Delta A\,\Delta B \geq \frac{1}{2}|\langle[A,B]\rangle|$

Commutation relations (angular momentum ones valid for spin S or orbital L):

$[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$

$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$, $[x, p] = i\hbar$

Energy of a particle in a magnetic field: $\boldsymbol{H} = -\boldsymbol{\mu}\cdot\boldsymbol{B} = -\frac{\boldsymbol{gq}}{\boldsymbol{2m}}\boldsymbol{S}\cdot\boldsymbol{B}$
(where g= gyromagnetic constant, it's about 2 for electrons with q=-e,
and **S** is the spin operator…)

Rabi's Formula: For a particle spin up in a B-field oriented at angle θ from the z-axis in the x-z plane, i.e. $\vec{B} = B_0\hat{z} + B_1\hat{x}$, we define $\omega_{0(1)} = \frac{e}{m}B_{0(1)}$
and find energies $E_\pm = \pm\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}$

$$\text{Prob(flipping + to −)} = \frac{\omega_1^2}{\omega_0^2+\omega_1^2}\sin^2\left(\frac{\sqrt{\omega_0^2+\omega_1^2}}{2}t\right) = \sin^2\theta\ \sin^2\left(\frac{E_+-E_-}{2\hbar}t\right)$$

EPR maximally entangled wavefunction is $|\psi\rangle = \frac{1}{\sqrt{2}}(\ |+\rangle_1|-\rangle_2 \quad - \quad |-\rangle_1|+\rangle_2\ )$,
which you showed in homework is also $|\psi\rangle = \frac{1}{\sqrt{2}}(\ |+\rangle_{1n}|-\rangle_{2n} \quad - \quad |-\rangle_{1n}|+\rangle_{2n}\ )$
(I.e. this state is anticorrelated in any spin direction $\hat{n}$.)

Spatial wavefunctions and the position representation:

States: $|\psi\rangle \doteq \psi(x) = \langle x|\psi\rangle$ Bras: $\langle\psi| \doteq \psi^*(x) = \langle\psi|x\rangle$

Operators: $\hat{\boldsymbol{x}} \doteq x$, $\hat{\boldsymbol{p}} \doteq -i\hbar\frac{\partial}{\partial x}$ Brackets: $\langle\varphi|\psi\rangle = \int_{-\infty}^{\infty}\varphi^*(x)\,\psi(x)dx$

Probability of finding a particle (a<x<b) = $\int_a^b |\psi(x)|^2dx$

Expectation values in a state ψ: $\langle A\rangle = \int_{-\infty}^{\infty}\psi^*(x)\hat{A}\,\psi(x)dx$

Schrödinger's time-independent equation: $\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right)\varphi_n(x) = E_n\,\varphi_n(x)$

Infinite Square well from 0 to a: Allowed energies $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$,
with corresponding normalized energy eigenfunctions $\varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$

Fourier Transforms: $\psi(x,t) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}\phi(p,t)dp$ is the Fourier transform of $\phi(p)$,
$\phi(p,t) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar}\psi(x,t)dx$ is the Inverse Fourier transform of $\psi(x)$. (And, $p = \hbar k$)

Heisenberg Uncertainty Principle: $\Delta x\,\Delta p \geq \hbar/2$

Angular momentum eigenfunctions:
$\boldsymbol{L}^2|l,m\rangle = l(l+1)\hbar^2\,|l,m\rangle$, $L_z|l,m\rangle = m\hbar\ |l,m\rangle\ \hbar$, with $|m| \leq l$.
In position space, $|l,m\rangle \doteq Y_l^m(\theta,\varphi)$

Schrodinger Equation in 3-D:

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{1}{\hbar^2r^2}\hat{L}^2\right]\psi(r,\theta,\varphi)+V(r)\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi),$$

where $\hat{L}^2 \doteq -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin\theta^2}\frac{\partial^2}{\partial\varphi^2}\right]$

Hamiltonian for particle on a ring: $H = \frac{L_z^2}{2I}$ with I=μr^2. **On a sphere:** $H = \frac{L^2}{2I}$

Hydrogen Atom: $|nlm\rangle \doteq R_{nl}(\boldsymbol{r})Y_l^m(\theta,\varphi)$, with $E_n = -13.6\ eV/n^2$,
Where n=1, 2, 3…∞, and $l = 0,1,2,\dots n-1$, and $m = -l,-l+1,\dots 0,\dots l-1,l$