3220 Notes Ch. 2 (SJP)

Now we get to the "formalism" of spin ½ QM. Our goal, as always, is to predict exp'tal outcomes!

Observables in QM are associated with OPERATORS

These act on a ket, transforming it to a new ket.

Notation: $\hat{A}|\psi\rangle = |\phi\rangle$

an operator — acts on a ket $\Rightarrow$ gives a ket.

(In vector space, an operator might be "rotate CW by 45°", or "flip across the x-axis" or "project out just the x-component",

Sometimes, you find special kets associated with a given operator which give back themselves, scaled by a number. These are called eigenkets, + the scaling # is the eigenvalue.

this must be an eigenstate!

$\hat{A}|\psi\rangle = a|\psi\rangle$ (I might rename $|\psi\rangle$ to be $|a\rangle$ to visually remind me of the value)

↳ a number

It's a Postulate (#2) of QM that observables are represented by operators.

3220 2.2 (SJP)

Operators have multiple distinct eigenvectors. Some will have a finite # (we'll deal with the infinite of "continuum" case later)

If you label these states' eigenvalues by a_n (an index) so $a_1 = 1^{st}$ eigenvalue, $a_2 = 2^{nd}$, up to $a_n = n^{th}$,

then $\hat{A}\,|a_n\rangle = a_n\,|a_n\rangle$

$\hookrightarrow$ a number, the eigenvalue!

a state labeled by n's eigenvalue

Postulate #3 of QM: If you measure the observable associated with operator $\hat{A}$, you will only measure one of the eigenvalues, a_n.

(you measure a number, the exp't's outcome!)

(Recall postulate #4 told us the probability of measuring a_n, which was $|\langle a_n|\Psi\rangle|^2$ if you start in state $|\Psi\rangle$)

Examples $\hat{S}_z$ is the operator associated with "measuring the z-component of spin", i.e. the outcome of a Z-S-G.

3220 2.3 (SJP)

So $\hat{S}_z |+\rangle_z = \frac{\hbar}{2} |+\rangle_z$ } This is our familiar Ket, $|+\rangle$, which is an eigenvector of S_z

→ This is the corresponding eigenvalue, the outcome.

$\hat{S}_z |-\rangle_z = -\frac{\hbar}{2} |-\rangle_z$

In ch.1 we introduced a matrix representation of Kets.

operators are also matrices in this representation.

Since $\hat{S}_z$ acts on a Ket + gives a Ket, and Kets are $\begin{pmatrix} \# \\ \# \end{pmatrix}$,

$\hat{S}_z$ must be a 2x2 matrix. In the $\hat{S}_z$ basis, we know

$|+\rangle_z \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle_z \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and so if $\hat{S}_z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

we get $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \hbar/2 \\ 0 \end{pmatrix}$ or $a = \hbar/2$, $c = 0$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -\hbar/2 \end{pmatrix}$, or $b = 0$, $d = -\hbar/2$

thus, in the $\hat{S}_z$ basis, we represent $\hat{S}_z$ as

→ this column refers to $|+\rangle$

→ " " " " $|-\rangle$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ this row refers to $\langle +|$

→ " " $\langle -|$

Notes: In the $\hat{S}_z$ basis, $\hat{S}_z$ is diagonal, eigenstates are "unit vectors"

3220 2.4

If $\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then note $\hat{A}|+\rangle \doteq \begin{pmatrix} a \\ c \end{pmatrix}$ and you can isolate "a" by doing this: $\langle +|\hat{A}|+\rangle = (1\ \ 0)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1\ \ 0)\begin{pmatrix} a \\ c \end{pmatrix} = a$

<u>act</u> the bra on the left

here's the $\langle +|$ bra

we call $\langle \text{bra}|\hat{A}|\text{ket}\rangle$ a "matrix element" for this reason.

$$\hat{A} = \begin{pmatrix} \langle +|\hat{A}|+\rangle & \langle +|\hat{A}|-\rangle \\ \langle -|\hat{A}|+\rangle & \langle -|\hat{A}|-\rangle \end{pmatrix}$$

convince yourself all 4 worked, as above.

Again, labeling rows + columns like I did on the prev. pages helps

$$\begin{array}{c|cc} \hat{A} & \text{column for } |+\rangle & \text{column for } |-\rangle \\ \hline \text{row for } \langle +| & \langle +|\hat{A}|+\rangle & \langle +|\hat{A}|-\rangle \\ \text{row for } \langle -| & \langle -|\hat{A}|+\rangle & \langle -|\hat{A}|-\rangle \end{array} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Standard math index notations

so $\boxed{\langle \underset{\text{row}}{n} | \hat{A} | \underset{\text{column}}{m} \rangle \leftarrow \text{a "matrix element"}}$

You can generalize this to "higher dimensionality" if e.g. you are working with spin-1 (3×3) matrices, etc...

3220 2.5

Eigenvalues + diagonalization

Suppose we know the operator's matrix representation. Can we deduce the e-values + e-vectors? Yes!

Suppose I'm given $\hat{A} \doteq \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and I seek a_n such that

$\hat{A}|a_n\rangle = a_n|a_n\rangle$ (Here, for a 2x2 matrix, I expect n=1 & 2...)

Write this out, with e.g. $|a_n\rangle \doteq \begin{pmatrix} c_{n1} \\ c_{n2} \end{pmatrix}$

So
$$\left.\begin{aligned} A_{11}c_{n1} + A_{12}c_{n2} &= a_n c_{n1} \\ A_{21}c_{n1} + A_{22}c_{n2} &= a_n c_{n2} \end{aligned}\right\}$$
Do it, convince yourself this is simply the line above, written out.

This is 2 eq'ns in 3 unknowns a_n, c_{n1}, and c_{n2}.

So
$$\left.\begin{aligned} (A_{11}-a_n)c_{n1} + A_{12}c_{n2} &= 0 \\ A_{21}c_{n1} + (A_{22}-a_n)c_{n2} &= 0 \end{aligned}\right\}$$
Homogeneous eq'ns.

Linear algebra proves you cannot solve these eq'ns simultaneously

unless
$$\begin{vmatrix} A_{11}-a_n & A_{12} \\ A_{21} & A_{22}-a_n \end{vmatrix} = 0.$$

[Arising from the fact that a matrix is invertible iff it has a non-zero det. (If det ≠ 0, the sol'n is "trivial")]

If we temporarily name our eigenvalue (a_n) as "λ", this says

$$|\hat{A} - \lambda\hat{I}| = 0, \quad \text{with } \hat{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

↳ "characteristic eq'n" for eigenvalue(s)

3220 2.6

For 2×2 matrices, our characteristic eq'n is 2^{nd} order in $\lambda \Rightarrow$ 2 roots.

Once you have them, you go back to your eigenvalue equation(s) to find c_{n1} and c_{n2}. (Because of the homogeneity, there will still be some ambiguity but you resolve it by normalizing + picking a simple overall phase)

Let's do some examples!

In general, we will face one of 2 general situations.

i) you know the eigenvalues + vectors, and seek the matrix.

ii) " " " matrix, + seek the e-values + e-vectors.

we just did situation (i) for $\hat{S}_z$ on the previous page. Let's do situation (ii) for $\hat{S}_z$: suppose I know $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What are the e-values + corresponding e-vectors?

The characteristic eq'n is $\left| \hat{S}_z - \lambda \hat{I} \right| = 0 = \begin{vmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{vmatrix}$

So $\left(\frac{\hbar}{2} - \lambda\right)\left(-\frac{\hbar}{2} - \lambda\right) - 0 = 0$, or $-\frac{\hbar^2}{4} + \lambda^2 = 0$, or $\lambda = \pm \hbar/2$.

Got the 2 eigenvalues! Let's solve for the "+" eigenvector:

So $\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$, or $\frac{\hbar}{2}\begin{pmatrix} a \\ -b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$.

The upper row is solved by <u>any</u> a. The lower row <u>requires</u> $b = 0$

So we have $\begin{pmatrix} a \\ 0 \end{pmatrix}$. Normalization gives $e^{i\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, + we <u>choose</u> $\theta = 0$ to get $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(you can work out that the e-vector for $\lambda_- = -\frac{\hbar}{2}$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, try it!)

Another example, $\hat{S}_y$. In Ch. 1, we found $|\pm_y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$

This is a "situation (c)" case, we know the eigenvectors, + e-values $= \pm\hbar/2$."

What is the matrix representation of $\hat{S}_y$? $\doteq \begin{pmatrix}a & b\\ c & d\end{pmatrix}$

Well... $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ \pm i/\sqrt{2}\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}1/\sqrt{2}\\ \pm i/\sqrt{2}\end{pmatrix}$

Evaluating just the top row using the "+" signs $\Rightarrow \frac{a}{\sqrt{2}} + \frac{ib}{\sqrt{2}} = +\frac{\hbar}{2}\cdot\frac{1}{\sqrt{2}}$

" " " " " " "−" " $\Rightarrow \frac{a}{\sqrt{2}} - \frac{ib}{\sqrt{2}} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}$

So $\left.\begin{array}{l} a+ib = +\hbar/2 \\ a-ib = -\hbar/2 \end{array}\right\}$ Adding $\Rightarrow 2a = 0 \Rightarrow a = 0$; subtracting $\Rightarrow 2ib = \hbar \Rightarrow b = -i\hbar/2$

Using bottom row gives $c + d$, + you get (check for yourself!)

$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ This is "in the S_z-basis", of course, because $|\pm\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix}$ already was.

[If I ask you "what is $\langle +|\hat{S}_y|+\rangle$ you can read it off $= 0$
or, $\langle +|\hat{S}_y|-\rangle = -i\hbar/2$. These are the "matrix elements"]
etc.

(Note: $\hat{S}_y$ is not diagonal in the S_z-basis.)

3220 2.8

What about situation ii: given $\hat{S}_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ can you find the eigenvalues + -vectors?

Let's do it: The characteristic eq'n is $|\hat{S}_y - \lambda\hat{I}| = 0 = \left|\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right|$

so $\begin{vmatrix} -\lambda & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\lambda \end{vmatrix} = 0 = \lambda^2 - \left(-i^2\frac{\hbar^2}{4}\right) = \lambda^2 - \frac{\hbar^2}{4}$, or $\lambda = \pm\hbar/2$ (Good!)

Let's find the eigenvector corresponding to $\lambda_- = -\hbar/2$ (Since McIntyre did the other!)

$\hat{S}_y\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$ or $\begin{pmatrix} -ib \\ ia \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

Both top + bottom rows give the same thing: $b = -ia$. So it's still ambiguous, but normalization saves us: $|a|^2 + |b|^2 = 1 = |a|^2 + |-ia|^2 = 2|a|^2$

So $a = 1/\sqrt{2}$ (choosing it Real + positive, as we can) + then $b = -ia = -i/\sqrt{2}$

So we found $|-\rangle_y \doteq \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (yup! We got back our familiar result.)

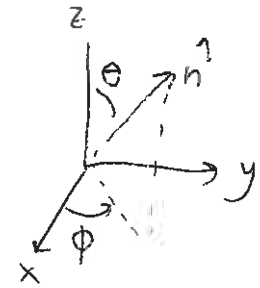
Summary: (for situation ii, given a matrix) You solve the characteristic eq'n to find the eigenvalues, then plug them in to find eigenvectors. This is called "diagonalizing" the matrix. (However, to really make it diagonal, There's another step, changing the basis. We won't bother with this)

McIntyre summarizes all Matrices, eigenvectors + eigenvalues for $S_{x,y,z}$

It's handy (see p. 41)

2.9

A ~~new~~ operator: What if we measure spin in the $\hat{n}$ direction?



$$\hat{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

So, the spin "in this direction" is $\vec{S}\cdot\hat{n}$

or $\hat{S}_n = \hat{S}_x \sin\theta\cos\phi + \hat{S}_y \sin\theta\sin\phi + \hat{S}_z \cos\theta$

We know the matrix representations for $\hat{S}_{x,y,\text{and } z}$, so we get

$$\hat{S}_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta[\cos\phi - i\sin\phi] \\ \sin\theta[\cos\phi + i\sin\phi] & -\cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{+i\phi} & -\cos\theta \end{pmatrix}$$

We are in a "situation ii": we know the operator's matrix, + wish to learn the eigenvalues (what will we <u>measure</u>?) + eigenvectors.

There is some algebra here, try it! First solve $\left| \hat{S}_n - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

and discover $\lambda = \pm\frac{\hbar}{2}$ (of course! Measuring spin in <u>any</u> direction gives $\pm\hbar/2$, it's spin-½!) And then solving $\hat{S}_n \begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$ gives

the eigenvectors, $|+\rangle_n = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}$ and $|-\rangle_n = \begin{pmatrix} \sin\theta/2 \\ -e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$

[Some <u>checks</u> for typos: In both, $|a|^2 + |b|^2 = 1$ ✓

also $\langle + | - \rangle = (\cos\frac{\theta}{2}\ \ e^{-i\phi}\sin\frac{\theta}{2}) \begin{pmatrix} \sin\theta/2 \\ -e^{i\phi}\cos\theta/2 \end{pmatrix} = \cos\frac{\theta}{2}\sin\theta/2\,(1-1) = 0$ ✓

($|+\rangle_n$ is a fully general ket; by choosing $\theta + \varphi$ you can get <u>any</u> ket in the spin ½ space. So all $|\psi\rangle$ states can be thought of as having a special <u>direction</u> that characterizes them)

2.10

Hermitian operators: Suppose $\hat{A}|\psi\rangle = |\phi\rangle$

Then $\langle\phi| \equiv \langle\psi|\hat{A}^{\dagger}$ ← that's a "dagger", the "Hermitian adjoint"

In general, it does not have to be the case that $\hat{A}^{\dagger} = \hat{A}$!

In linear algebra, you prove $\hat{A}^{\dagger} = (\hat{A}^{T})^{*}$, the complex conjugate of
(see bottom of page)
the transpose of $\hat{A}$. So, there are matrices for which $\hat{A}^{\dagger} = \hat{A}$,

+ we call them "Hermitian"

Ex: Is $\hat{S}_y$ Hermitian? $\hat{S}_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{S}_y^{T} \xrightarrow{\text{transpose! Flip row/column}} = \frac{\hbar}{2}\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}$

so $(\hat{S}_y^{T})^{*} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \hat{S}_y$ yes!

[In QM, all operators that correspond to physical observables are Hermitian
This is sometimes included in the statement of Postulate #2]

Formally, suppose $\hat{A}|\psi\rangle = |\phi\rangle$ so $\langle\phi| = \langle\psi|\hat{A}^{\dagger}$

Let's let both sides act on an arbitrary state $|\beta\rangle$, so

$\langle\phi|\beta\rangle = \langle\psi|\hat{A}^{\dagger}|\beta\rangle$ But we showed in Ch.1 $\langle\phi|\beta\rangle = \langle\beta|\phi\rangle^{*}$

so $\langle\psi|\hat{A}^{\dagger}|\beta\rangle = \langle\beta|\phi\rangle^{*} = (\langle\beta|\hat{A}|\psi\rangle)^{*}$

Ah! That's how you find matrix elements of $\hat{A}^{\dagger}$, you
can see that the matrix for $A^{\dagger}$ is the complex conjugate of the transpose of $\hat{A}$,
(As claimed above) [Linear Algebra also says eigenvalues of Hermitian matrices
are real. Good! Measurables must be real!]

SJP Notes 2.11

Projection Operators:

In 2-d vector space, we can "project" a vector onto the x-axis, graphically

This is the projection

In QM, we can construct a projection operator, $\hat{P}$, that does something similar.

1st, Recall the inner product $\langle \text{bra} | \text{ket} \rangle \rightarrow$ a #!

Now, Let's define an outer product $|\text{ket}\rangle\langle \text{bra}| \rightarrow$ an operator.

Example: If $|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$ and $|\phi\rangle \doteq \begin{pmatrix} c \\ d \end{pmatrix}$

then $\langle \psi | \phi \rangle = (a^* \ b^*) \begin{pmatrix} c \\ d \end{pmatrix} = \underbrace{a^* c + b^* d}_{\text{a \#!}}$

But $|\psi\rangle\langle\phi| \doteq \begin{pmatrix} a \\ b \end{pmatrix} (c^* \ d^*) = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$ an operator!

It's a 2x2 matrix, an operator.

If you act this on a ket, you get a (new) ket.

SJP Notes 2.12

<u>Ex</u>: What is $|+\rangle\langle+|$?

It's $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

<u>Ex</u>: what is $|-\rangle\langle-| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

In general, $|\psi\rangle\langle\psi|$ is a "projection operator".

Let's see why! Consider $\hat{P}_+ \equiv$ "Projection onto +Z spin"

Define it as $\hat{P}_+ \equiv |+\rangle\langle+|$

what does it do to some state $|\psi\rangle \doteq \begin{pmatrix}a\\b\end{pmatrix}$?

or $|\psi\rangle = a|+\rangle + b|-\rangle$

equivalently

(see Top of page example!)

$\hat{P}_+|\psi\rangle = \underbrace{|+\rangle\langle+|}_{\text{this is } \hat{P}_+}\psi\rangle$

In <u>matrix</u> language, $\hat{P}_+|\psi\rangle \doteq \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\0\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix}$

oh! $\hat{P}_+$ "projects out" <u>just</u> the upper component, the "$+\rangle_z$" <u>piece</u> of the state!

Let's redo that in pure bra-ket notation:

If $|\psi\rangle = a|+\rangle + b|-\rangle$

just refocus your eyes!

$$\underbrace{\hat{P}_+}_{\text{operator}} \underbrace{|\psi\rangle}_{\text{ket}} = \Big(\underbrace{|+\rangle\langle+|}_{\text{operator}}\Big)\underbrace{|\psi\rangle}_{\text{ket}} = \underbrace{|+\rangle}_{\text{ket}} \underbrace{\langle+|\psi\rangle}_{\text{number}}$$

$$= \underbrace{\langle+|\psi\rangle}_{\#} \underbrace{|+\rangle}_{\text{ket}}$$

(you can move #'s to either side of a ket!)

$$= \Big(\langle+| \underbrace{a|+\rangle + b|-\rangle}_{\text{just write out } |\psi\rangle}\Big)|+\rangle = \Big(a\langle+|+\rangle + b\langle+|-\rangle\Big)|+\rangle$$

#'s come out of brackets

$$= a|+\rangle$$ because $\langle+|+\rangle = 1$, $\langle+|-\rangle = 0$

<u>Same</u> result as previous page.

$\hat{P}_+$ <u>acts</u> on a superposition of $a|+\rangle + b|-\rangle$ and yield <u>just</u> the "spin up" <u>part</u>, $a|+\rangle$

It "projects out" the "+z component" of the state.

SJP Notes 2.14

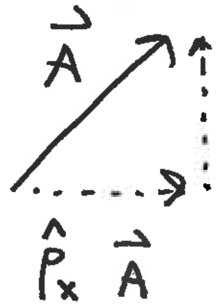
Note that $|+\rangle$ and $|-\rangle$ are a complete basis (for spin $\frac{1}{2}$)

and since $\hat{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ & $\hat{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\hat{P}_+ + \hat{P}_- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \hat{\mathbb{1}}$

This is no accident. If you "project onto the $|+\rangle$ direction" + add (superpose) that with "projecting " " $|-\rangle$ direction" you get back what you started with.

So $\hat{P}_+ + \hat{P}_- = \hat{\mathbb{1}}$ is also called "completeness".

In our 2-D vector analogy, adding the two projections into the 2 basis directions



$\hat{P}_y \vec{A}$ = Projection of $\vec{A}$ in the $\hat{y}$ direction

gives you back what you start with: $(\hat{P}_x + \hat{P}_y)\vec{A} = \hat{\mathbb{1}}\vec{A}$
$= \vec{A}$.

$\hat{\mathbb{1}} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ = "the identity matrix",

$$\hat{\mathbb{1}}\hat{A} = \hat{A}\hat{\mathbb{1}} = \hat{A}$$

SJP Notes 2.15

Example What is $\hat{P}_- |+\rangle$.

I think I expect "nothing", since $|+\rangle$ has no "down-ness"

Let's see: $\hat{P}_- |+\rangle = \underbrace{\left(|-\rangle\langle -|\right)}_{\text{the } \hat{P}_- \text{ operator, by def}} |+\rangle = |-\rangle \langle -|+\rangle$

↳ Just stare, drop the parentheses which merely guide the eye

But $\langle -|+\rangle = 0$, so yes

$\hat{P}_- |+\rangle = 0.$ → (the "zero-ket")

In matrix notation, $\hat{P}_- |+\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Recall Postulate # 4 says $\text{Prob}\left(\begin{matrix}\text{measuring } +\hbar/2 \\ \underline{\text{given}}\ |\psi\rangle\end{matrix}\right) = |\langle +|\psi\rangle|^2$

Write this as $\text{Prob}(+) = |\langle +|\psi\rangle|^2 = \langle +|\psi\rangle\langle +|\psi\rangle^*$

↳ Def of "square magnitude"

$= \underbrace{\langle +|\psi\rangle}_{a\ \#}\underbrace{\langle \psi|+\rangle}_{a\ \#}$ ⟷ (Because $\langle \phi|\psi\rangle = \langle \psi|\phi\rangle^*$ always)

$= \langle \psi|+\rangle\langle +|\psi\rangle$ ⟷ Because you switch order of #'s

$= \langle \psi\left(|+\rangle\langle +|\right)\psi\rangle$ ⟷ Add parentheses to guide the eye

$\text{Prob}(+) = \langle \psi|\hat{P}_+|\psi\rangle$ A useful + general result!

(give $|\psi\rangle$)

SJP Notes 2.16

Why are projection operators important in QM?

If you measure a component of spin on an input state $|\psi\rangle$, the result (and the output state) are in general unpredictable, only the probabilities are predictable.

But you will get some result if you measure (!), and you will end up in the eigenstate corresponding to the eigenvalue you measured.

You "collapse" the state, into one of its projections!

This is the "measurement" or "collapse" postulate, stated formally on the next page. But in words:

If you measure $\hat{A}$ on state $|\psi\rangle$ and get a result a_n,
(you always measure one of the eigenvalues of $\hat{A}$, postulate 3!)
(This happens with a probability given by postulate 4!)
Then you will end up in a new state $|\psi_{new}\rangle$, which is the (normalized) projection of $|\psi\rangle$ in the "a_n" direction.

SJP Notes 2.17

Postulate #5 : Given $|\psi\rangle$, after measuring $\hat{A}$ and getting a_n, the new (normalized) output state is

$$|\psi_{new}\rangle = \frac{\hat{P}_n |\psi\rangle}{\sqrt{\langle\psi|\hat{P}_n|\psi\rangle}}$$

$\hat{P}_n$ → this is the projection operator for outcome a_n ($= |a_n\rangle\langle a_n|$)

→ The whole numerator is the projection of $|\psi\rangle$ in the "a_n-direction" (a ket!)

this denominator is a #, it normalized the final state.

Note from p.15, then $\langle\psi|P_n|\psi\rangle$ = "Probability of getting outcome a_n"

Ex : Start in $|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$, and measure $\hat{S}_z$

[redacted] Prob ($S_z \Rightarrow +\hbar/2$) $= |a|^2$, because [redacted]

Prob (see p.15!) $= \langle\psi|P_+|\psi\rangle = (a^* \; b^*) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$

$= (a^* \; b^*) \begin{pmatrix} a \\ 0 \end{pmatrix} = |a|^2$, Good, that's postulate 4!

Say we do the measurement + happen to get $+\hbar/2$. What's my new state? Post. 5 says

$$|\psi_{new}\rangle = \frac{\hat{P}_+|\psi\rangle}{\sqrt{\langle\psi|P_+|\psi\rangle}} \doteq \frac{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}{\sqrt{|a|^2}} = \frac{\begin{pmatrix} a \\ 0 \end{pmatrix}}{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Yes!! (Normalized spin up!)

Ex: Start in $|\psi\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$, and measure $\hat{S}_x$.

$\text{Prob}(S_x \text{ comes out } +\hbar/2) = \langle\psi|\hat{P}_{+x}|\psi\rangle$ ⟷ Bottom of p. 15

where $\hat{P}_{+x} \equiv |+\rangle_x\,{}_x\langle +|$ (Projects onto "x+")

So **Prob** $= (a^* \; b^*)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}(1 \; 1)\begin{pmatrix} a \\ b \end{pmatrix} = (a^* \; b^*)\cdot\underbrace{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\text{This is } \hat{P}_{+x}\,!}\begin{pmatrix} a \\ b \end{pmatrix}$

$$= \frac{1}{2}(a^* \; b^*)\begin{pmatrix} a+b \\ a+b \end{pmatrix} = \frac{1}{2}\left(|a|^2 + a^*b + b^*a + |b|^2\right)$$

$$= \frac{1}{2}|a+b|^2$$ ⟷ convince yourself

Now, say we measure S_x + happen to get $+\hbar/2$

What's our new output state? Post 5 says:

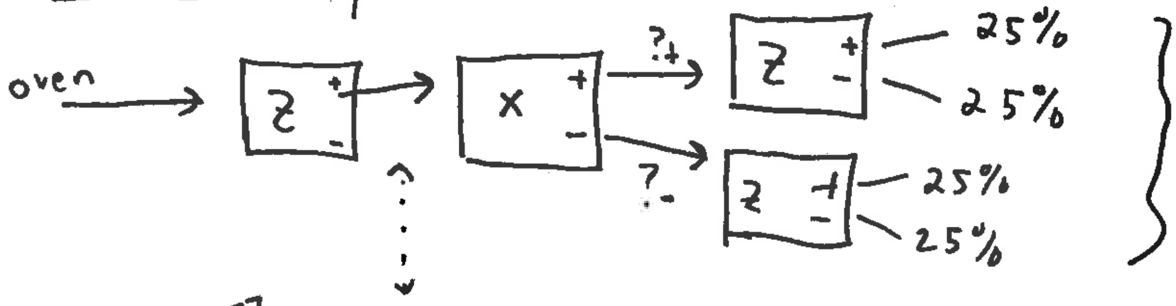
$$|\psi_{new}\rangle = \frac{\hat{P}_{+x}|\psi\rangle}{\sqrt{\langle\psi|P_{+x}|\psi\rangle}} = \frac{\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}}{\sqrt{\frac{1}{2}|a+b|^2}} = \frac{1}{\sqrt{2}}\frac{\begin{pmatrix} a+b \\ a+b \end{pmatrix}}{|a+b|}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle_x$$

[Oh, of course!! If I measured S_x and got $+\hbar/2$, I must end up in the state $|+\rangle_x$!]

SJP Notes 2.19

<u>Revisit</u> Exp'ts 4a, b, (+c) with the formalism!



summary of Exp'ts 4a+b

After 1^{st} S-G we know our upper beam is $|+\rangle$

what's the state I labeled "$?_+$" above? It's the <u>state</u> you get if you <u>measure</u> S_x and get $+\hbar/2$, by post 5

$$|?_+\rangle = \frac{\hat{P}_{+x}|+\rangle}{\sqrt{\langle +|\hat{P}_{+x}|+\rangle}}$$

= I <u>just</u> computed this on prev page! Set $a=1$, $b=0$, you get $|+_x\rangle$!

The prob<u>abil</u>ity of this outcome is $\langle +|\hat{P}_{+x}|+\rangle = \frac{1}{2}|a+b|^2$

$= \frac{1}{2}$ here, yes!!

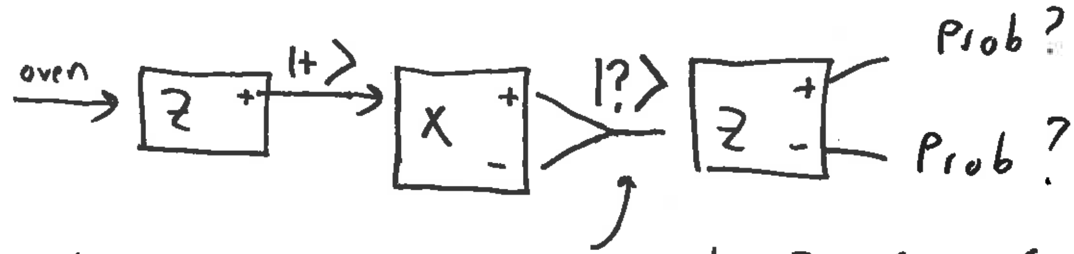
$$|?_-\rangle = \frac{\hat{P}_{-x}|+\rangle}{\sqrt{\langle +|\hat{P}_{-x}|+\rangle}} = \frac{\overbrace{|-_x\rangle\langle -_x|}^{\text{this is } \hat{P}_{-x}}+\rangle}{\sqrt{\langle +\underbrace{|-_x\rangle\langle -_x|}_{\text{again, } \hat{P}_{-x}}+\rangle}} \doteq \frac{\frac{1}{\sqrt{2}}(1\ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{\sqrt{\cdots\cdots}}$$

$$= \frac{\overbrace{\frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}}^{\text{this is } \hat{P}_{-x}}\begin{pmatrix}1\\0\end{pmatrix}}{\sqrt{(1\ 0)\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\-\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}} = \frac{\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}}{\sqrt{\frac{1}{2}\cdot 1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = |-\rangle_x$$

oh of course! (And, with probability $\rightarrow \frac{1}{2}$, yes!)

SJP Notes 2.20

Exp't 4C



What is $|?\rangle$, the input state to the final Z - Stern Gerlach?

Post 5 says $= |\psi_{out}\rangle = \dfrac{(\hat{P}_{+x} + \hat{P}_{-x})|+\rangle}{\sqrt{\langle +| \quad " \quad |+\rangle}}$ But,

That numerator is simply $\mathbb{1}$, by completeness!

so $|\psi_{out}\rangle = |+\rangle$, (as we saw to our surprise) in Ch. 1.

100% chance of exiting with spin z $\Rightarrow +\hbar/2$.

See McIntyre to work it out in detail, if you want.
It's good "bRA-Ket" practice.

Comment: If we observe the particles after the X-device, before combining them, we made another measurement (!) + so we collapsed the wave function(!) entering the final Z S-G to be either $|+_x\rangle$ or $|-_x\rangle$, whichever we observed, with 50/50 odds

But whichever it is, that's not the superposition state above, + now we get 50% odds of $+\hbar/2$ for the last outcome, Very different if you "observe"!

Measurement + averages: Given a collection of identically prepared quantum states $|\Psi\rangle$, although individual measurements of operators might be probabilistic, we can still compute averages over many measurements ≡ "Expectation Values".

$$\langle \hat{A} \rangle \equiv \text{"expectation value of } \hat{A} \text{" given a state } |\Psi\rangle \equiv \langle \Psi | \hat{A} | \Psi \rangle$$

(I'll show why on the next page)

Ex: Suppose $|\Psi\rangle = |+\rangle$, what is $\langle \hat{S}_x \rangle$?

Answer: $\langle \hat{S}_x \rangle = \langle + | \hat{S}_x | + \rangle \doteq (1\ 0) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$

$= \frac{\hbar}{2} (1\ 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. Does this make sense? Yes!

This is like Exp't 2: I expect half the x-measurements to give $+\hbar/2$ and half to give $-\hbar/2$, the average $= 0$

Note: You never measure 0! It's not the "expected outcome"!

[Like tossing a coin with "1" on one side + "0" on the other. The average outcome = "expectation value" = 0.5, but no single coin toss ever gives you that!

SJP Notes 2.22

Here's the math: $\langle \hat{A} \rangle \equiv \langle \Psi | \hat{A} | \Psi \rangle = \langle \Psi | \hat{A} \cdot \hat{\mathbb{1}} | \Psi \rangle$

(because $\hat{A} \cdot \hat{\mathbb{1}}$ is the same as $\hat{A}$)

But completeness says $\hat{\mathbb{1}} = \sum_n | \Psi_n \rangle \langle \Psi_n |$

where $|\Psi_n\rangle$ represents the complete set of basis states of $\hat{A}$,

with $\hat{A} | \Psi_n \rangle = a_n | \Psi_n \rangle$

↳ the "n-th eigenvalue"

so $\langle \hat{A} \rangle = \langle \Psi | \hat{A} \underbrace{\sum_n | \Psi_n \rangle \langle \Psi_n |}_{\text{this is } \hat{\mathbb{1}}} \Psi \rangle$

$$= \sum_n \langle \Psi | \hat{A} | \Psi_n \rangle \langle \Psi_n | \Psi \rangle$$ (sums come out of brackets)

#'s come out too!

$$= \sum_n \langle \Psi | a_n | \Psi_n \rangle \langle \Psi_n | \Psi \rangle = \sum_n a_n \underbrace{\langle \Psi | \Psi_n \rangle}_{\#} \cdot \underbrace{\langle \Psi_n | \Psi \rangle}_{\#}$$

$$= \sum_n a_n | \langle \Psi_n | \Psi \rangle |^2 = \sum_n a_n \cdot \text{Prob (measuring } a_n)$$

this is Postulate 4

Ah, yes, the average is the sum of "outcome * Prob (outcome)"

Uncertainty or "Standard deviation" or "RMS deviation"

These are names for the spread in outcomes, symbolically ΔA

Note: ΔA is not the "average deviation from expectation"

because that would be $\langle(\hat{A} - \langle\hat{A}\rangle)\rangle = \langle\hat{A}\rangle - \langle\hat{A}\rangle = 0$

The average of # is that # !

(you are as likely to deviate up as down)

Instead, we will define

$$\Delta A \equiv \sqrt{\langle(A - \langle A\rangle)^2\rangle}$$ = "Square root of average of squared deviations"

= Root - Mean - Square !

$$= \sqrt{\langle(\hat{A}^2 - 2\hat{A}\underbrace{\langle\hat{A}\rangle}_{\text{a \#}} + \underbrace{\langle A\rangle^2}_{\text{a \#}})\rangle}$$

→ again, average of #'s is that # !

$$= \sqrt{\langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle\langle\hat{A}\rangle + \langle\hat{A}^2\rangle}$$

$$= \sqrt{\langle\hat{A}^2\rangle - \langle A\rangle^2}$$

← this is a formula you may have used in Phy 1140 or 2150 with classical measurements!

SJP Notes 2.24

Ex: Suppose my state $|\psi\rangle = |+\rangle$. What is ΔS_x?

(Think about, try to guess before calculating. I expect to measure $+\hbar/2$ and $-\hbar/2$ with equal prob, I expect a spread!)

Here $\langle S_x^2 \rangle = \langle + | \hat{S}_x \hat{S}_x | + \rangle$

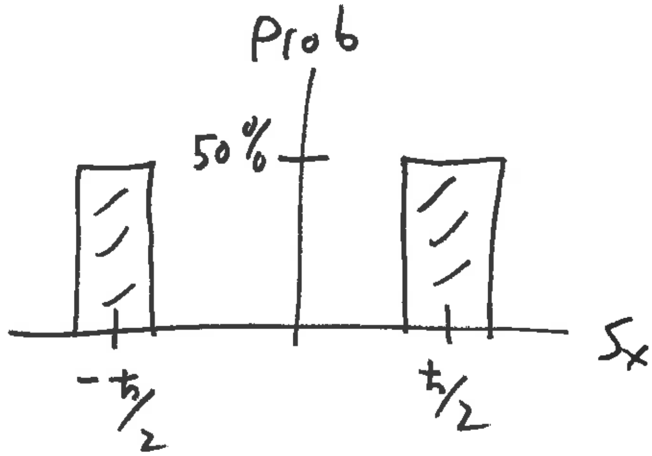
$$\doteq (1\ 0)\ \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar^2/4$$

While $\langle S_x \rangle = 0$ (I did this on p. 21)

so $\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\hbar^2/4 - 0} = \hbar/2$.

Indeed!

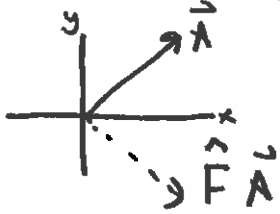


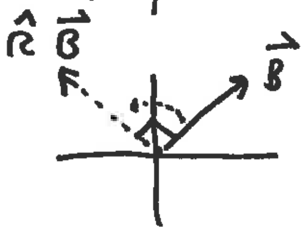
Average is 0, but $\Delta S_x = \hbar/2$, (it's $0 \pm \hbar/2$!)

SJP Notes 2.25

Commutation: The order of operator matters! Consider e.g.

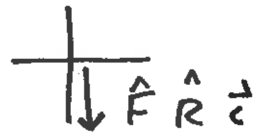
in 2-D vector space $\hat{F}$ = "flip around x-axis" + $\hat{R}$ = "rotate 90° CCW"

so 

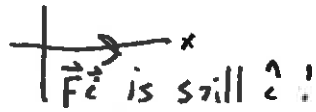
and 

. Now consider

$\hat{F}\hat{R}\vec{i} = \hat{F}$ 

$=$ 

— Not the

$\hat{R}\hat{F}\vec{i} = \hat{R}$ 

$=$ 

— same!!

Order matters

Define $[\hat{A}, \hat{B}]$ ≡ "commutator of $\hat{A}$ and $\hat{B}$"

$\equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ (a new operator)

If order does not matter, if $\hat{A}\hat{B}$(on anything) $=$ $\hat{B}\hat{A}$(on that same thing)

then $\hat{A}\hat{B} = \hat{B}\hat{A}$, and $[\hat{A}, \hat{B}] = \hat{O}$,

in such a case "$\hat{A}$ and $\hat{B}$ commute"

This is nice, because (proven in book!) then eigenstates of $\hat{A}$ are also eigenstates of $\hat{B}$, + so measurement of $\hat{A}$ does not alter or mess of measurement of $\hat{B}$.

SJP Notes 2.26

If $[\hat{A},\hat{B}]=0$, you can simultaneously know the values of measurements of either/both A and B (if you are in a (common) eigenstate.

We say "commuting observables" are compatible, measuring A doesn't alter the state measured by B.

But if operators do not commute, they do not have common "bases", + measuring one messes up the other. You cannot then simultaneously know both outcomes for one given starting state!

$\hat{S}_x$, $\hat{S}_y$, or $\hat{S}_z$ do not commute!

$$\boxed{\begin{aligned} [S_x, S_y] &= i\hbar S_z \neq 0 \\ [S_y, S_z] &= i\hbar S_x \neq 0 \\ [S_z, S_x] &= i\hbar S_y \neq 0! \end{aligned}}$$

One example: $[S_x, S_y] = S_xS_y - S_yS_x$

$$= \frac{\hbar}{2}\cdot\frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{\hbar}{2}\cdot\frac{\hbar}{2}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)$$

$$= \hbar\cdot i\cdot\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar S_z, \text{ as claimed.}$$

[Like we saw in ch. 1: you cannot know S_x and S_z simultaneously. Measuring 1 messes up the other!]

SJP Notes 2.27

Uncertainty Principle

Theorem (Proven in some books, it's tedious but followable!)

For any operators $\hat{A}$ and $\hat{B}$ and any given state $|\psi\rangle$,

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

These are quantum uncertainties. The randomness of QM is manifest!

Ex: Suppose $[\hat{A}, \hat{B}] = 0$, they commute. Then $\Delta A \, \Delta B \geq 0$

Greater than or equal. You could have a state $|\psi\rangle$ with uncertainty in A (and B), but you could also " " " " " where both A and B are known simultaneously, i.e. $\Delta A = \Delta B = 0$

that agrees with the theorem, $0 \geq 0$.

Ex: $|\psi\rangle = |+\rangle$, then $\Delta S_x = \hbar/2$, + our theorem says since $[S_x, S_x] = 0$ (obviously, I hope!) then $(\Delta S_x)^2 \geq 0$.

yes, $(\hbar/2)^2 \geq 0$.

If $|\psi\rangle = |+\rangle_x$, then $\Delta S_x = 0$, + still $(\Delta S_x)^2 = 0 \geq 0$.

SJP Notes 2.28

If 2 operators do not commute, this inequality is interesting!

Ex: $[S_x, S_y] = i\hbar S_z$ (from notes p. 26)

so $\Delta S_x \, \Delta S_y \geq \frac{1}{2} \left| \langle i\hbar S_z \rangle \right| = \frac{\hbar}{2} \left| \langle S_z \rangle \right|$

↳ #'s come out, and $|i| = 1$

If $|\psi\rangle = |+_x\rangle$, then (can you convince yourself?) $\langle S_z \rangle = 0$,

+ in this case $\Delta S_x \, \Delta S_y \geq \frac{\hbar}{2} \cdot 0 = 0$.

well, that's not so interesting. Any 2 non-neg things multiplied will be ≥ 0. In this example, $\Delta S_x = 0$, so $0 \geq 0$ .... OK...

But!! If $|\psi\rangle = |+\rangle$, then $\langle S_z \rangle = +\hbar/2$,

and now $\Delta S_x \, \Delta S_y \geq \frac{\hbar}{2} \cdot \hbar/2 > 0$

↳ Not "or equals to"!!

Ooh! you cannot have either ΔS_x or $\Delta S_y = 0$ here!

Both must be uncertain! This is QM; If you know S_z, you cannot know either S_x or S_y!

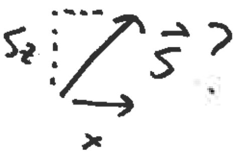
The measurements are "incompatible".

SJP Notes p.29

when I tell you $S_z = +\hbar/2$, you might try to picture a classical top spinning with $\vec{S}$ pointing up in z.

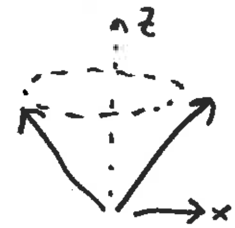
But no, because then I know S_x and S_y (both are 0)

Or, you might try picturing a tilted top, but again,



you cannot have a known x or y component!

Spin is not a classical vector!



I tend to picture a "fuzzy cone" with known S_z but fuzzy (uncertain) S_x + S_y ...

B.T.W., the famous application of this theorem is

$\Delta \hat{x} \, \Delta \hat{p} \geq \hbar/2$, but we haven't yet talked about $\hat{x}$ or $\hat{p}$ operators, which are continuous variables, so we'll come back to it. But it arises from the same idea!

Consider the $\hat{S}^2$ operator $\equiv \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$

you can write down all 3 as matrices + work it out!

$$\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \hat{\mathbb{1}}$$

Since $\hat{\mathbb{1}}$ commutes with anything, $\hat{\mathbb{1}}\hat{A} = \hat{A}\hat{\mathbb{1}} = \hat{A}$,

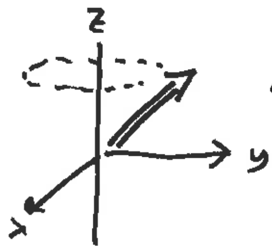
$$[\hat{S}^2, \hat{S}_x] = 0 = [S^2, S_y] = [S^2, S_z]$$

So you <u>can</u> simultaneously know S_z and S^2!

For any spin $\frac{1}{2}$ state, $S^2|\psi\rangle = \frac{3}{4}\hbar^2 \mathbb{1}|\psi\rangle = \frac{3}{4}\hbar^2|\psi\rangle$

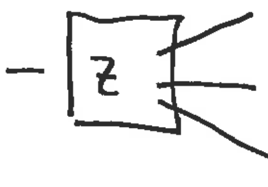
So all spin $\frac{1}{2}$ particles have S^2 measured as $\frac{3}{4}\hbar^2$

So my "fuzzy" picture is
(<u>If</u> $|\psi\rangle = |+\rangle$) $\Rrightarrow$



My spin vector is a fuzzy cone with definite z-component $\hbar/2$ + definite <u>length</u> $\sqrt{\frac{3\hbar^2}{4}} = \sqrt{3}\,\hbar/2$

What about Spin-1? These are particles giving 3 spots in a Z-Stein Gerlach



state $|1\rangle$, eigenvalue of $S_z = +\hbar$

state $|0\rangle \longrightarrow 0$

state $|-1\rangle \longrightarrow -\hbar$.

In this "Z-basis", $|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $|0\rangle \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|-\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

and $\hat{S}_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

All the formal results from before follow, you just use 3×3 matrices + 3×1 kets for computing!

Experiment (just like for spin ½ in ch.1) will determine $\hat{S}_x$ and $\hat{S}_y$ (and you can find <u>their</u> eigenvectors + eigenvalues. See MacIntyre!)

The "machinery" is then the same; Calculate probabilities use $\text{Prob}(a_n) = |\langle \psi_n | \psi \rangle|^2$, and calculate expectation Values of Δ's, or final states using 3×3 matrices...

It's the <u>same</u> math, just slightly more tedious matrix algebra

If you go to higher spin, it's still the same, just larger-dimension matrices. For spin s, we will explicitly label states $|s, m\rangle$

$s = \frac{1}{2}$ for spin $\frac{1}{2}$
$= 1$ " " 1,
etc

this is the z-component outcome. We used to only mention it, because we knew $s = \frac{1}{2}$!

~~So e.g our old state $|+\rangle$ is now $|\frac{1}{2}, \frac{1}{2}\rangle$~~

the convention here is

$$\hat{S}^2 |s, m\rangle = s(s+1)\hbar^2 |s\ m\rangle$$

so e.g for spin $\frac{1}{2}$, $\frac{1}{2}\left(\frac{3}{2}\right)\hbar^2 = \frac{3}{4}\hbar^2$, as we found!

and $\hat{S}_z |s, m\rangle = m\hbar \quad |s\ m\rangle$

So e.g our old friend $|+\rangle$ is now written $|\frac{1}{2}, +\frac{1}{2}\rangle$

Indeed, you can generalize beyond spin. Any system with an operator with n eigenvalues can be represented with $n \times n$ matrices, + all the rules + formalism + postulates are unchanged. (Only the algebra involves $n \times n$ matrices to compute)