Time evolution: Physics should predict the future!

e.g. $\vec{F} = d\vec{p}/dt$, ... In QM, time dependence is postulated

$$\boxed{i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H}(t)\, |\Psi(t)\rangle}$$

← Post #6

The Schrödinger eq'n!

[TDSE = Time-dependent Schrod. Eq'n]

Here $|\Psi(t)\rangle$ is the time dependent state.

$\hat{H}$ = the classical Hamiltonian, the "energy operator"

(It's the observable associated with energy.)

Of course $\hat{H} = \hat{H}^\dagger$, it must be Hermitian ↔ observable.

Why? It's a postulate! (Motivated by Classical Hamilton's eq'ns)

Operators have eigenvectors & eigenvalues, we often label an eigenvector by the values, so we say

$$\hat{H}\, \underbrace{|E_n\rangle}_{\text{An eigenvector (or "eigenstate") of } \hat{H}} = \underbrace{E_n}_{\text{this is the eigenvalue, the ENERGY of this ket}} \, |E_n\rangle$$

→ the label for the eigenvector is the outcome

This eq'n is called the time independent Schrodinger eq'n.

or TISE

$\hat{H}$ can depend explicitly on time, but often does not.

we start (+ focus on) with time-<u>ind</u>ependent $\hat{H}$'s.

But even then, <u>states</u> have time dependence, they "evolve".

The eigenvectors of $\hat{H}$ are a <u>complete</u> <u>basis</u>

we often <u>choose</u> this as our basis, call it the "energy basis"

Completeness says <u>ANY</u> $|\Psi(t)\rangle = \sum_n \underbrace{C_n(t)}_{\text{this is where any/all time-dependence sits}} \underbrace{|E_n\rangle}_{\text{time-independent, if } \hat{H} \text{ is}}$

"Orthonormal" tells us $\langle E_n | E_m \rangle = \delta_{nm} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$

Different systems have different $\hat{H}$ $\Rightarrow$ different <u>basis</u>

you <u>choose</u> a basis, to make calculation easy!

Let's figure out those $C_n(t)$'s by plugging "completeness",

$|\Psi(t)\rangle = \sum_n C_n(t) \, |E_n\rangle$, <u>into</u> the full TIME-DEPENDENT

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} \; |\Psi(t)\rangle$$ Schrodinger-eqn

3-3

a pure #

so $$i\hbar \frac{\partial}{dt}\left(\sum_n C_n(t)\,|E_n\rangle\right) = \hat{H}\left(\sum_n C_n(t)|E_n\rangle\right)$$

No time dependence (if $\hat{H}$ has no ")

can slide #'s past operators

also, $\hat{H}$ (sum) = sum $\hat{H}$

No time deriv of states

$$\therefore\ i\hbar \sum_n \frac{dC_n(t)}{dt}|E_n\rangle = \sum_n C_n(t)\ \hat{H}|E_n\rangle$$

Recall, TISE!

$$= \sum_n C_n(t)\ E_n|E_n\rangle$$

I claim each individual term in the sum on the left side matches with " " " " " " " " right side.

Proof! Hit both sides with one term, e.g. $\langle E_K|$

you may choose any K you want,

then $$i\hbar \sum_n \frac{dC_n(t)}{dt}\langle E_K|E_n\rangle = \sum_n C_n(t)\,E_n\ \langle E_K|E_n\rangle$$

these are #'s, so $\langle E_K|$ slides past!

(Ditto)

But $\langle E_K|E_n\rangle = \delta_{Kn}$, meaning all terms in those sums vanish except the one term where n = your K

For that one term, $$i\hbar \frac{d}{dt}C_K(t) = E_K C_K(t)$$

So $i\hbar \frac{d\, C_K(t)}{dt} = C_K(t)\, E_K$, a 1^{st} order linear ODE for $C_K(t)$

I am familiar with this ODE (are you?) the sol'n is just

$$C_K(t) = C_K(0)\, e^{i E_K t/\hbar}$$

↳ there is one "unknown constant" in 1^{st} order ODE'S

$C_K(0)$ is the initial condition

At $t=0$, suppose I know our starting state $|\Psi(t=0)\rangle$

Then completeness says $|\Psi(0)\rangle = \sum_n C_n(0)\, |E_n\rangle$

But I can use this to find any $C_K(0)$ I need, again by

hitting left side with $\langle E_K|$, giving

$$\langle E_K | \Psi(0)\rangle = \sum_n C_n(0) \underbrace{\langle E_K | E_n\rangle}_{\delta_{Kn} = \begin{cases} 0 & \text{if } n \neq K \\ 1 & \text{if } n = K \end{cases}} = C_K(0)$$

So Given $|\Psi(0)\rangle$ I know every $C_K(0)$!

as time goes by, " " " $C_K(t) = \left(e^{i E_K t/\hbar}\right) \cdot C_K(0)$

(I may drop the "$(t=0)$" notation, + just call $C_K(0) \equiv C_K$)

If no time-dependence is shown, assume it's $t=0$

Bottom line, the <u>full</u> sol'n to the T.D.S.E. is

$$\boxed{|\Psi(t)\rangle = \sum_n C_n\, e^{-iE_nt/\hbar}\, |E_n\rangle}$$

(* of course, we are Assuming $\hat{H}$ is time-<u>independent</u>...)

where $C_n \equiv \langle E_n|\Psi(t=0)\rangle$ are "initial condition" constants

If you know your <u>basis</u> states $|E_n\rangle$, + your starting state, these are simple "braket" computations, algebra!

So the general path to "predicting the future" in QM is:

a) you must know the Hamiltonian. That's "the physics of your system" (but, it's classical, by Hypothesis)

b) you must find the eigenvalues (E_n) + eigenvectors $(|E_n\rangle)$ of your $\hat{H}$. That's called "diagonalizing the Hamiltonian" It's a linear-algebra problem, if we are dealing with spins

c) you need a starting condition, $|\Psi(t=0)\rangle$.

<u>Then</u>: find $C_n = \langle E_n|\Psi(t=0)\rangle$

+ the top of the page gives you $|\Psi(\text{any later time})\rangle$!

If you measure any<u>thing</u>, say $\hat{A}$ (with known eigenvalues a_j + known eigenvectors $|a_j\rangle$) then Postulate 4 says

$$\text{Prob}(\text{measuring } a_j) = |\langle a_j|\Psi(t)\rangle|^2$$

So I can predict <u>all</u> exp'tal outcomes at <u>all</u> times (probabilistically)

That's basically all of QM!!

Example: Let's start @ $t=0$ in an eigen<u>state</u> of Ener<u>gy</u>,

I pick "2", could be <u>any</u> state

so say $|\Psi(0)\rangle = |E_2\rangle$

then $\langle E_n|\Psi(0)\rangle = \langle E_n|E_2\rangle = \delta_{n2}$, so $C_n = \delta_{n2}$

In other words, $C_1 = 0$, $C_2 = 1$, $C_3 = C_4 = C_5 = \ldots = 0$

+ Now $|\Psi(t)\rangle = \sum_n \underset{\sim}{C_n} e^{-iE_n t/\hbar} |E_n\rangle = e^{-iE_2 t/\hbar} |E_2\rangle$

almost <u>all</u> terms vanish except

Not much going on here. We start in $|E_2\rangle$, + as time goes by, there's just an <u>overall</u> phase out front.

That has <u>no</u> observable effects, so we call energy eigenstates "Stationary States", no time-dependence of <u>observables</u>.

What about <u>other</u> observables, $\hat{A}$?

$$\text{Prob}(\text{getting } A_K) = |\langle a_K | \Psi(t)\rangle|^2$$

$$= |\langle a_K | \underbrace{e^{iE_2 t/\hbar}}_{\text{just a \#}} |E_2\rangle|^2 = |e^{iE_2 t/\hbar}|^2 \, |\langle a_K | E_2\rangle|$$

pull it out

$$= |a_K | E_2\rangle|^2$$

So, as claimed, this probability doesn't depend on time.

All observables remain "time independent" for this

<u>pure</u> energy sta<u>tion</u>ary <u>state</u>.

But more interesting physics happens if you start in

a <u>superposition</u> of energy states.

Consider $|\Psi(0)\rangle = a|E_1\rangle + b|E_2\rangle$

By inspection (do you agree??) $c_1 = \langle E_1 | \Psi(0)\rangle = a$

$c_2 = b$.

and our <u>full</u> time dependent state will be...

$$|\psi(t)\rangle = \underbrace{a e^{-iE_1 t/\hbar}}_{\text{right? this is } c_1(t)} |E_1\rangle + \underbrace{b e^{-iE_2 t/\hbar}}_{\text{this is } c_2(t)} |E_2\rangle$$

Note: overall phase doesn't matter, but relative phase can!

Measurement of Energy: (Energy is "special"!)

what's Prob(measuring E_2, say) $= |\langle E_2|\psi(t)\rangle|^2$

$$= |a e^{-iE_1 t/\hbar} \underbrace{\langle E_2|E_1\rangle}_{0\text{, by orthogonality}} + b e^{-iE_2 t/\hbar} \underbrace{\langle E_2|E_2\rangle}_{1\text{, normal!}}|^2$$

$= |b|^2.$

Similarly, Prob $(E_1) = |a|^2.$

These are time-independent constants!

So Energy measurements are also "stationary", time-independent

— Q.M. doesn't (grossly) violate conservation of energy (yay!)

But, energy isspecial! What if we measure some other observable, $\hat{A}$? If $[\hat{A}, \hat{H}] = 0$, the basis is common

+ Prob(measuring A_n) is also stationary!

But what if $[\hat{A}, \hat{H}] \neq 0$, no "common basis". So <u>now</u>

$$\text{Prob}(a_1 \text{ measurement}) = |\langle a_1 | \Psi(t)\rangle|^2$$

Now, since "bases are <u>not</u> common", but $|E_n\rangle$ is <u>complete</u>, it <u>must</u> be true that $|a_1\rangle = \alpha_1 |E_1\rangle + \alpha_2 |E_2\rangle$

some constants!

$$\text{so Prob}(a_1) = \left|\left(\alpha_1^* \langle E_1| + \alpha_2^* \langle E_2|\right)\left(a e^{-\frac{iE_1 t}{\hbar}}|E_1\rangle + b e^{-\frac{iE_2 t}{\hbar}}|E_2\rangle\right)\right|^2$$

$$= \left|\alpha_1^* a e^{-iE_1 t/\hbar} \underbrace{\langle E_1|E_1\rangle}_{1} + \alpha_2^* b e^{-iE_2 t/\hbar} \underbrace{\langle E_2|E_2\rangle}_{1}\right|^2$$

$$= \underbrace{\left|e^{-iE_1 t/\hbar}\right|^2}_{\text{factor out overall phase; this is } = 1} \underbrace{\left|\alpha_1^* a + \alpha_2^* b e^{-i(E_2 - E_1)t/\hbar}\right|^2}_{\text{oh! there is a time dependence now}}$$

$$= |\alpha_1^* a|^2 + |\alpha_2^* b|^2 + 2\,\text{Re}\underbrace{\left(\alpha_1^* a \alpha_2^* b e^{-i(E_2 - E_1)t/\hbar}\right)}_{}$$

some <u>sinusoidal</u> variation,

at frequency $\frac{E_2 - E_1}{\hbar}$

(With <u>more</u> than 2 Energies superposed you get <u>more</u> cross terms @ different frequencies, a "Fourier" situation...)

Rest of Ch. 3 is more examples:

① Given or find $\hat{H}$ ② Diagonalize ③ Solve for observables ...

Ex: Spin $\frac{1}{2}$ object in a $\vec{B}$-field.

Recall (Day 1!) we say classical $P.E. = -\vec{\mu} \cdot \vec{B}$

Let's neglect KE for now (it "sits" in the field)

so $\hat{H} = -g \cdot \frac{q}{2m_e} \vec{S} \cdot \vec{B}$ (again, from 1st day of class)

$= +\frac{e}{m} \vec{S} \cdot \vec{B}$ for electrons (or Ag atoms ...) with $g = 2$ and $q = -e$.

This is the game. Postulate 6 says use the classical $\hat{H}$.

If $\vec{B} = B_0 \hat{z}$ the math is the simplest, let's start there.

So $\hat{H} = \frac{e}{m} B_0 S_z \equiv \omega_0 S_z$ defining $\omega_0 = \frac{e B_0}{m_e}$

Note: $\hat{H} \propto \hat{S}_z$, so "energy basis" is same as "S_z-basis".

$$\hat{H} = \omega_0 S_z = \omega_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{in this basis}$$

we're done with step ①, we have our $\hat{H}$.

Continuing: I already know the eigenvalues $(\pm \hbar\omega_0/2)$

and the corresponding eigenvectors $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$

(Since $[\hat{H}, \hat{S}_z] = 0$ this is a <u>common</u> basis for both)

In ket notation, $\hat{H}\,|+\rangle = +\frac{\hbar\omega_0}{2}|+\rangle \equiv E_+\,|+\rangle$

this is the energy of a spin-up electron

$\hat{H}\,|-\rangle = -\frac{\hbar\omega_0}{2}|-\rangle = E_-\,|-\rangle$

we're done with step ②, we have basis states of $\hat{H}$.

(Here, there are only <u>2</u> possible energy eigenstates)

Now we need an initial condition.

If I give you $|\Psi(0)\rangle = C_+\,|+\rangle + C_-\,|-\rangle$

These 2 #'s specify our <u>starting</u> spin

then $|\Psi(t)\rangle = C_+ e^{-iE_+t/\hbar}|+\rangle + C_- e^{-iE_-t/\hbar}|-\rangle$

and we've solved for the state at <u>future</u> times.

(<u>Recall</u> $E_\pm \equiv \pm \frac{\omega_0 \hbar}{2} = \pm \frac{\hbar}{2} \cdot \frac{eB_0}{m_e}$ are givens)

(Note that spin ↑ ⇒ E_+ ⇒ <u>higher</u> energy. (Because $q_e = -e$))

If we start "spin-up", then $C_1 = 1$, $C_2 = 0$

$|\psi(t)\rangle = e^{-i\frac{\omega_0 t}{2}} |+\rangle$ Just an overall phase.

Prob (measure $E_+ = \frac{\hbar\omega_0}{2}$) $= |\langle E_+|\psi(t)\rangle|^2 = 1$ for all time

Prob (measure spin up) $= |\langle +|\psi(t)\rangle|^2 = 1$ " " "

Nothing really happens!

classically, "spin up" in a B-field $\Rightarrow \vec{\tau} = \vec{\mu} \times \vec{B} = 0$.

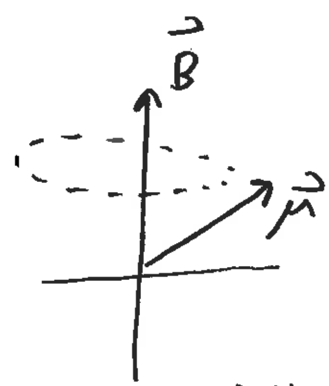
If we start in $a|+\rangle + b|-\rangle$,

$$\left.\begin{aligned} \text{Prob } (E_+) &= |a|^2 \\ \text{Prob } (\text{spin up}) &= |a|^2 \end{aligned}\right\} \text{ for all time.}$$

So any spin state is "stationary" here... with respect to Energy of S_z measurements.

But what about other measurements?

Let's investigate!



Classically, $\vec{\mu}$ "precesses",

S_z is constant but

S_x, S_y oscillate

McIntyre solves the general case. I'm going to do a specific ex:

Let $|\psi(0)\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle \doteq \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$

By inspection, $c_+ = 3/5$, $c_- = 4/5$, and so

$$|\psi(t)\rangle = \frac{3}{5} e^{-iE_1 t/\hbar} |+\rangle + \frac{4}{5} e^{-iE_2 t/\hbar} |-\rangle$$

$$= \begin{pmatrix} \frac{3}{5} e^{-i\frac{\omega_0 t}{2}} \\ \frac{4}{5} e^{+i\frac{\omega_0 t}{2}} \end{pmatrix}$$

$$\text{Prob (spin z-up)} = |\langle + | \psi(t)\rangle|^2 = \left| (1 \; 0) \begin{pmatrix} \frac{3}{5}e^{-\cdots} \\ \frac{4}{5}e^{+\cdots} \end{pmatrix} \right|^2$$

$= \frac{9}{25}$, independent of time.

What about "Expectation value" of S_z?

$$\langle S_z \rangle = \langle \psi(t) | S_z | \psi(t) \rangle = \frac{\hbar}{2} \left(\frac{3}{5} e^{+i\cdots}, \; \frac{4}{5} e^{-i\cdots} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{5}e^{-\cdots} \\ \frac{4}{5}e^{+\cdots} \end{pmatrix}$$

Note, conjugates!)

$= \frac{\hbar}{2}\left(\frac{9}{25} - \frac{16}{25}\right) = \frac{\hbar}{2}\left(\frac{-7}{25}\right)$ independent of time

(See McIntyre's general sol'n, this is in agreement)

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But what about S_x? $\text{Prob}(+x) = |{}_x\langle +|\psi(t)\rangle|^2$

$$= \left|\frac{1}{\sqrt{2}}(1,1)\begin{pmatrix}\frac{3}{5}e^{-i\omega_0 t/2}\\ \frac{4}{5}e^{+i\omega_0 t/2}\end{pmatrix}\right|^2 = \frac{1}{2}\left|\frac{3}{5}e^{-i\frac{\omega_0 t}{2}} + \frac{4}{5}e^{+i\frac{\omega_0 t}{2}}\right|^2$$

$$= \frac{1}{2}\cdot\underbrace{\left|e^{-i\frac{\omega_0 t}{2}}\right|^2}\left|\frac{3}{5}+\frac{4}{5}e^{i\omega_0 t}\right|^2$$

I Pulled out a common overall phase

This is the relative phase, not the factor 2 change

$$= \frac{1}{2}\cdot\frac{1}{25}\left(3+4e^{i\omega_0 t}\right)\left(3+4e^{-i\omega_0 t}\right) = \frac{1}{50}\left(9+16+2\cdot 12\cdot\cos\omega_0 t\right)$$

$$= \frac{1}{2}+\frac{24}{50}\cos\omega t$$ Oscillating probability, like our precessing top would suggest!

C.F. McIntyre, here my $\frac{\theta}{2} = \cos^{-1}\left(\frac{3}{5}\right) = 53^\circ$, my $\varphi = 0$.

How about $\langle S_x\rangle$? Use $S_x = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$, + get

$$\langle S_x\rangle = \frac{\hbar}{2}\left(\frac{3}{5}e^{+i\omega_0 t/2}, \frac{4}{5}e^{-i\cdots}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{3}{5}e^{-i\omega_0 t/2}\\ \frac{4}{5}e^{+i\omega_0 t/2}\end{pmatrix}$$

complex conj for bra!

$$= \frac{\hbar}{2}\left(\frac{3}{5}e^{+i\frac{\omega_0 t}{2}}, \frac{4}{5}e^{-i\cdots}\right)\begin{pmatrix}\frac{4}{5}e^{+i\frac{\omega_0 t}{2}}\\ \frac{3}{5}e^{-i\cdots}\end{pmatrix} = \frac{\hbar}{2}\cdot\frac{12}{25}\left(e^{+i\omega_0 t}+e^{-i\omega_0 t}\right)$$

$$= \frac{\hbar}{2}\cdot\frac{24}{25}\cos\omega_0 t$$ again, oscillates

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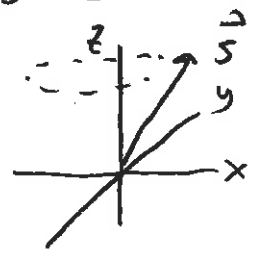
Pictorial summary



The classical + quantum pictures fit... S_z is fixed, but S_x + S_y precesses.

Called "Larmor precession", at Larmor frequency $\omega_0 = e B_0 / m$

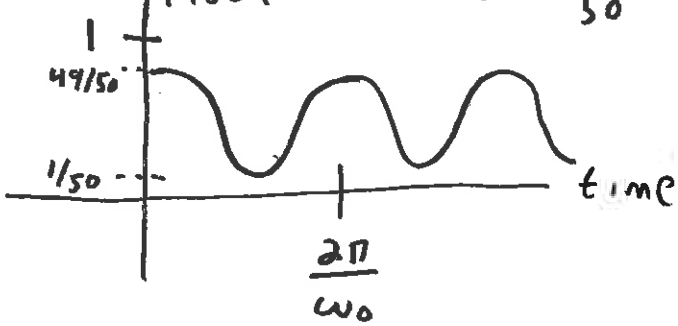
Here, the Expectation values all behave perfectly classically

This is "Ehrenfests theorem", + is more general!

This calculation is of practical value: precessing spins in $\vec{B}$-fields are the physics of MRI scanners, + much more.

$\text{Prob}(\text{measure } +x) = \frac{1}{50}(25 + 24 \cos \omega t)$

In my example



We never get to "100%" $-x$, nor "0%" in x.

We started with a "z-component" $\langle S_z \rangle = -7\hbar/50$

And this never goes away, so we will never get to a "pure X" state.

(Only starting in $|\pm\rangle_x$ will give you back a pure $|\pm\rangle_x$ state at a later time)

What if I gave you $|\Psi(t=0)\rangle = \frac{3}{5}|+\rangle_x + \frac{4}{5}|-\rangle_x$

Note!

you may be tempted to guess $|\Psi(t)\rangle$ = this formula

with $e^{\pm iEt/\hbar}$ factors stuck in each term, next to "$\frac{3}{5}$" + "$\frac{4}{5}$"

But no, you can't do that!

$|\pm\rangle_x$ are not eigenvectors of our $\hat{H}$!

- The procedure is clear, you must first rewrite your starting state as $a|+\rangle + b|-\rangle$

(i.e, "expand in your energy basis" first)

and then you are free to stick in $e^{iE_\pm t/\hbar}$ factors next to "a" and "b"! That's the time-dependence game

What about $\vec{B}$ that is <u>not</u> just pure $\hat{z}$.

E.g $\vec{B} = B_0 \hat{z} + B_1 \hat{x}$ $\leftarrow$ we can choose our $\hat{x}$ so this is true...

$$\hat{H} = -\vec{\mu} \cdot \vec{B} = +\frac{ge}{2m_e}\left(\hat{S}_z B_0 + \hat{S}_x B_1\right) \equiv \omega_0 \hat{S}_z + \omega_1 \hat{S}_x$$

$$\boxed{\text{Defining } \omega_0 = \frac{e}{m_e} B_0 \quad \omega_1 = \frac{e}{m_e} B_1}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 \\ \omega_1 & -\omega_0 \end{pmatrix}$$ $\leftarrow$ Using $\hat{S}_x$ + $\hat{S}_z$ matrices from earlier

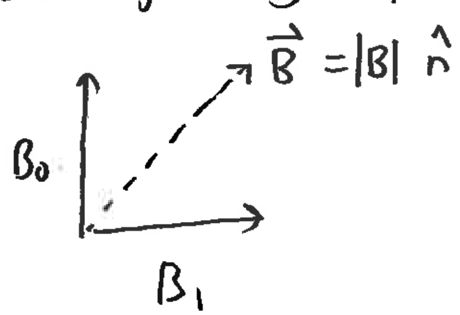
Step ① done!

② We need to <u>diagonalize</u> this $\hat{H}$: find eigenvalues + eigenvectors

It's a 2x2 matrix, just do it! I leave it as an exercise:

<u>Eigenvalues turn out to be $\pm\frac{\hbar}{2}\sqrt{\omega_0^2 + \omega_1^2} \equiv \pm\frac{\hbar}{2}\bar{\omega}$</u>

Some geometry helps here: $\vec{B}$ points in some $\hat{n}$ direction!



By inspection <u>of this figure</u>: $|B| = \sqrt{B_0^2 + B_1^2} \equiv \frac{m}{e}\bar{\omega}$

$$\tan\theta = B_1/B_0 = \omega_1/\omega_0$$

$$\sin\theta = \frac{B_1}{\sqrt{B_0^2 + B_1^2}} = \omega_1/\bar{\omega}$$

$$\cos\theta = B_0/\sqrt{B_0^2 + B_1^2} = \omega_0/\bar{\omega}$$

so $\omega_0 = \bar{\omega}\cos\theta$, $\omega_1 = \bar{\omega}\sin\theta$ and $\hat{H} = \frac{\hbar}{2}\bar{\omega}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

and $E_{\pm} = \pm\frac{\hbar}{2}\bar{\omega}$

Oh! That matrix is familiar, it is ~~$\bar{w}$~~ $\bar{w}$ · the S_n matrix, with $\varphi = 0$ (← that's because we picked $B_y = 0$!)

But we did the linear algebra for this already!

$$|+\rangle_n = \cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle$$ ← since $\phi = 0$

$$|-\rangle_n = +\sin\frac{\theta}{2}\,|+\rangle - \cos\frac{\theta}{2}\,|-\rangle$$

we know the energy eigenvectors! So, done with step ② !

Step ③: I need to know the starting state.

Ex what if $|\Psi(0)\rangle = |+\rangle$ what happens!

We have to "expand this state in the energy basis" !

i.e. I must write $|+\rangle = a\,\underbrace{|+\rangle_n}_{\text{energy state}} + b\,\underbrace{|-\rangle_n}_{\text{Energy state!}}$

How to find a or b? Easy! "Dot" both sides ....

e.g. to get a, hit on left with ${}_n\langle+|$, to give

$${}_n\langle+|+\rangle = a\,\underbrace{{}_n\langle+|+\rangle_n}_{1!} + b\,\underbrace{{}_n\langle+|-\rangle_n}_{0\text{ by orthogonality}} = a$$

Easy enough, from top of page, this is $\left(\cos\frac{\theta}{2}\langle+| + \sin\frac{\theta}{2}\langle-|\right)|+\rangle$

$= \cos\frac{\theta}{2} = a$ Got it!

Similarly, to get "b", hit left with ${}_n\langle -|$, to get

$$\underset{\sim\sim\sim}{{}_n\langle -|+\rangle} = a\,{}_n\langle -|+\rangle_n + b\,{}_n\langle -|-\rangle_n = b$$

But from top of prev page, $\underline{\underline{\text{this}}}$ is $\sin \theta/2$.

So our $|\psi(0)\rangle = a|+\rangle_n + b|-\rangle_n$ ← these are energy eigenstate

so now $|\psi(t)\rangle = a e^{-iE_+ t/\hbar} |+\rangle_n + b e^{-iE_- t/\hbar} |-\rangle_n$

$$= \cos\frac{\theta}{2}\, e^{-iE_+ t/\hbar} |+\rangle_n + \frac{\sin\theta}{2} e^{-iE_- t/\hbar} |-\rangle_n$$

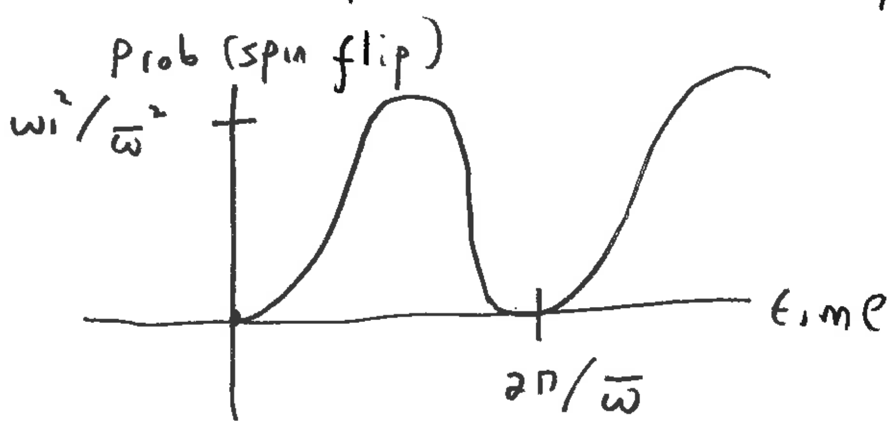
$\underline{\text{Recall}}$: $E_\pm = \pm\frac{\hbar}{2}\bar{\omega} = \pm\frac{\hbar}{2}\sqrt{\omega_0^2+\omega_1^2}$, + $(\theta = \tan^{-1} \omega_1/\omega_0)$

Now we can compute whatever observa$\underline{\text{ble}}$ we want.

E.g. Prob(Sz is down) $= |\langle -|\psi(t)\rangle|^2$

Do the algebra! we have $|\psi(t)\rangle$ above. It's a bit of complex algebra, result is Prob (Sz is ↓) $= \dfrac{\omega_1^2}{\bar{\omega}^2} \sin^2\left(\dfrac{\bar{\omega} t}{2}\right)$

This is Rabi's formula. we started spin up, + $\underline{\text{flip}}$ with this probability



Discussion:

- If $\omega_1 = 0$ ($\vec{B}$ is pure $\hat{z}$) then $\bar{\omega} = \omega_0$, + we get no flips

  we saw that result before!

- If $\omega_0 = 0$ ($\vec{B}$ is pure $\hat{x}$, $\perp$ to our starting spin) then $\bar{\omega} = \omega_1$

  and $\text{Prob}(\text{flip}) = \sin^2 \omega_1 t/2$

  Ah yes! This is a rotated version of another result from before, where we found a pure-x state rotates around the z-axis / B-field at the Larmor frequency. Here we have effectively the same thing, a pure-z states rotated around the x-axis / B-field!

- If $\omega_0 >> \omega_1$, a "small perturbation" in the x direction then $\omega_1^2/\bar{\omega}^2 << 1$, $\bar{\omega} \simeq \omega_0$, so it has small prob of flipping

  at freq $\approx \omega_0$

There are many QM systems, not just "spins", where there are basically 2 states available. (E.g atoms with 2 low energies, or neutrinos with 2 "flavors")

If $\hat{H} = \underbrace{E_0}_{\text{something!}} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ then we have basically solved this system!

(as often happens)

Example: For "2-flavor neutrinos", the "energy basis" + "flavor basis" are different. Here, for a given momentum p, there are two energies, $E_i = \sqrt{p^2c^2 + m_i^2 c^4}$ with $i = 1$ or 2

Here neutrino $|\nu_1\rangle$ has mass m_1
$|\nu_2\rangle$ " " m_2] this is the energy basis

But electron neutrinos are a mix, $|\nu_e\rangle = \cos\frac{\theta}{2}|\nu_1\rangle + \sin\frac{\theta}{2}|\nu_2\rangle$

muon " " orthogonal, $|\nu_\mu\rangle = +\sin\frac{\theta}{2}|\nu_1\rangle - \cos\frac{\theta}{2}|\nu_2\rangle$

The physics is total different, but the math + formalism matches what we just did: ~~from previous page.~~

$$|\psi(t)\rangle = \cos\frac{\theta}{2}\, e^{-iE_1 t/\hbar}|\nu_1\rangle + \sin\frac{\theta}{2}\, e^{-iE_2 t/\hbar}|\nu_2\rangle$$

and $\text{Prob(flavor flips)} = \frac{\omega_1^2}{\bar{\omega}^2}\sin^2\frac{\bar{\omega}t}{2}$

where now $\underbrace{E_- - E_+}_{= \hbar\bar{\omega}}$ will be replaced by $E_2 - E_1$.
$\equiv \hbar\bar{\omega}$ Defines our $\bar{\omega}$.

(and $\frac{\omega_1}{\bar{\omega}} = \sin\theta$, as before)

See p. 17.

Last topic of ch. 3 is a special case of time-dependent Hamiltonian. Alas, we must start from scratch, much of the last pages had assumed $\hat{H}$ was time independent

Suppose $\vec{B} = B_1 \cos\omega t \; \hat{i} + B_1 \sin\omega t \; \hat{j} + B_0 \hat{z}$

Like our last example, but the field oscillates — this is as before

so $\hat{H} = -\vec{\mu}\cdot\vec{B} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{+i\omega t} & -\omega_0 \end{pmatrix}$

again, like before, but we added in x and y oscillations

$\omega_0 \equiv \frac{eB_0}{m} \qquad \omega_1 \equiv \frac{eB_1}{m}$

we must solve $i\hbar \frac{d}{dt}|\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$

with $|\Psi(t)\rangle = \begin{pmatrix} C_+(t) \\ C_-(t) \end{pmatrix}$

so $i\hbar \begin{pmatrix} \dot{C}_+ \\ \dot{C}_- \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{+i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} C_+ \\ C_- \end{pmatrix}$

This is two coupled ordinary 1st order ODE's.

Suppose $\omega = \omega_0$ (we "drive" our perturbation at the Larmor freq)

then define a new quantity $\alpha_+(t) = e^{+i\frac{\omega t}{2}} C_+(t)$, $\alpha_-(t) = e^{-i\omega t/2} C_-(t)$] Let $\omega = \omega_0$ for simplicity

so $C_+ = e^{-i\omega_0 t/2}\alpha_+(t)$, $C_- = e^{+i\omega_0 t/2}\alpha_-(t)$

$$\dot{C}_+ = \frac{dC_+}{dt} = \dot{\alpha}_+ e^{-i\omega_0 t/2} - i\frac{\omega_0}{2} e^{-i\frac{\omega_0 t}{2}}\alpha_+$$

$$\dot{C}_- = \dot{\alpha}_- e^{+i\omega_0 t/2} + i\frac{\omega_0}{2} e^{+i\frac{\omega_0 t}{2}}\alpha_-$$

$$i\hbar\dot{C}_+ = \frac{\hbar}{2}\left(\omega_0 C_+ + \omega_1 e^{-i\omega_0 t} C_-\right)$$

common!

$$\Rightarrow i\hbar\left(\dot{\alpha}_+ e^{-i\omega_0 t/2} - i\frac{\omega_0}{2} e^{-i\frac{\omega_0 t}{2}}\alpha_+\right) = \frac{\hbar}{2}\begin{pmatrix}\omega_0 e^{-i\omega_0 t/2}\alpha_+ \\ +\omega_1 e^{-i\omega_0 t/2}\alpha_-\end{pmatrix}$$

so $\dot{\alpha}_+ = -\frac{i}{2}\omega_1\alpha_-$ and

$$i\hbar\dot{C}_- = \frac{\hbar}{2}\left(\omega_1 e^{i\omega_0 t} C_+ - \omega_0 C_-\right) = \frac{\hbar}{2}\begin{pmatrix}\omega_1 e^{i\omega_0 t/2}\alpha_+ \\ -\omega_0 e^{i\omega_0 t/2}\alpha_-\end{pmatrix}$$

$$i\hbar\left(\dot{\alpha}_- e^{i\omega_0 t/2} + i\frac{\omega_0}{2} e^{i\frac{\omega_0 t}{2}}\alpha_-\right)$$

common

so $\dot{\alpha}_- = -\frac{i}{2}\omega_1\alpha_+$

$$\text{so } i\hbar\frac{d}{dt}\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix} = \frac{\hbar\omega_1}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}$$

That's the same math we saw before, where we had

$$i\hbar \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \hat{H}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \quad \text{with} \quad \hat{H} = \frac{\hbar\omega_1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It's the "B-field is $B_1\hat{x}$, spin starts pure up"

we had $|+\rangle_n = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ since $\theta = 90°$ here

$|-\rangle_n = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$.

and we had $|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_+t/\hbar}|+\rangle_n + \frac{1}{\sqrt{2}}e^{-iE_-t/\hbar}|-\rangle_n$

(where I assumed we start in $|+\rangle$ state, so $a = b = 1/\sqrt{2}$

so $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = e^{-iE_+t/\hbar}\begin{pmatrix} \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \end{pmatrix} + e^{-iE_-t/\hbar}\begin{pmatrix} \frac{1}{2} \\ 1/2 \end{pmatrix}$

or $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-i\omega_1 t/2} + e^{i\omega_1 t/2} \\ e^{-i\omega_1 t/2} - e^{+i\omega_1 t/2} \end{pmatrix} = \begin{pmatrix} \cos \omega_1 t/2 \\ -i \sin \omega_1 t/2 \end{pmatrix}$

But this time, these are α_+ + α_- we're finding so

$$\alpha_-(t) = -i \sin \omega_1 t/2 = C(t)\, e^{-i\omega t/2}$$

Thus $|C_-(t)|^2 = \sin^2 \omega_1 t/2$

3-25

Discussion: When we drive the (tiny!) B_1 term at $\omega = \omega_0$ we get 100% spin flip probability at frequence $\omega_1/2$.

Spin flips absorb energy, (we take energy from rotating field)

If you go "up to down" this lowers energy, emission other way, absorption.

(where "up" $\Rightarrow$ higher state in Energy $\Rightarrow E_+$)

- We can stop the perturbative field after $T = \pi/\omega_1$
  this is a π-pulse, we flipped the spins.
- we can detect energy flows $\Rightarrow$ detect presence of spins. (MRI)

  we can drive off resonance - McIntyre does the math.
  (Prob of spin flips falls off fast with freq.)
- EM waves have oscillating fields, math is again same, + this is how light can absorb or cause emission.