Quantum Mechanics
Website: www.colorado.edu/physics/phys3220

Please check it out – for resources, notes, up-to-date contact information, grading and exam information (evening exams, note the dates!) office hours, and lots more!

Note - we have a "co-seminar" this term, Phys32210. It meets right after class every Friday. It's optional, pass/fail, and has no homework! (Because of that, alas it doesn't technically count towards your physics or CU degree) It's a chance to wrestle with conceptual and computational underpinnings every week. Please join in whenever you can, you are always welcome even if you do not register for it.

Our text is McIntyre's "Quantum Mechanics". It takes what is known as a "spins-first" approach. More on why this is a good thing to come shortly! But, if you would like a more traditional textbook (which may prove useful, especially later in the term, as a supplement), I can recommend Griffiths' Quantum Mechanics text as well. If you continue on in physics, you will likely end up with a pile of QM books on your shelf – every author has a unique perspective, and more than any other branch of physics, these very different approaches are helpful in making sense of quantum physics

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Introductory Remarks, some background, perspectives, and motivations!

Q.M. is a new (well, ok – it's 100+ years old, but new compared to Newton) and absolutely necessary way of predicting the behavior of microscopic objects.
It is also the best-tested, experimentally-validated physical theory known to humankind.

It is based on several radical, and generally also counter-intuitive, ideas/observations:
1) Many aspects of the world are fundamentally probabilistic, not deterministic.
2) Some aspects of the world are essentially discontinuous ("quantized")

Bohr famously said: "*Those who are not shocked when they first come across quantum theory cannot possibly have understood it.".* But, shocking or not, it's an accurate, predictive, descriptive theory of the world. Although pop culture suggests there's something fuzzy or metaphysical about it, we will find that it is a rigorous theory, with well-defined mathematical and logical underpinnings that allow us to calculate and predict observables. And, we (you!) can and will develop *intuitions* and skills for using it to do and learn useful physics (and engineering, for that matter). That's our task for the term.

Humans have divided physics into a few artificial categories, called theories, such as

- classical mechanics (non-relativistic and relativistic)
- electricity & magnetism (classical version)
- quantum mechanics ("QM") (non-relativistic)
- general relativity (theory of gravity)
- thermodynamics and statistical mechanics
- quantum electrodynamics and quantum chromodynamics (relativistic versions of QM)

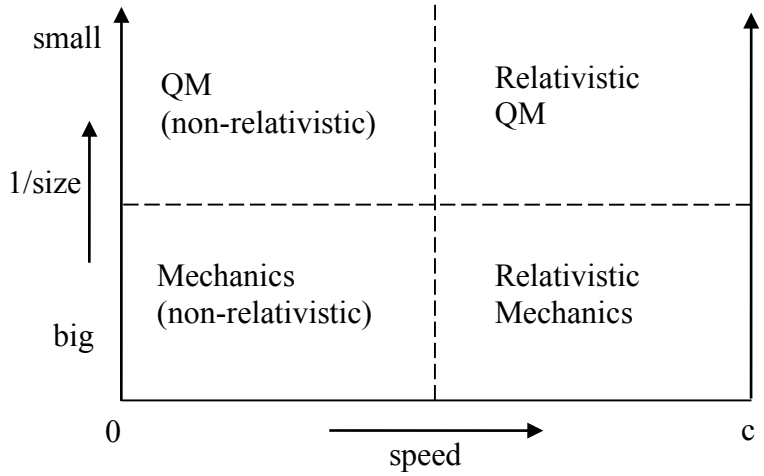
Each of these theories can be taught without much reference to the others. This is a bad way to view physics, of course, since we live in a single universe that must obey one set of rules. Our goal should be to look for the connections between apparently different topics. We can only really learn a concept by seeing it in context, that is, by answering the question: how does this new concept fit in with other, previously learned, concepts?

Each of these theories must rest on a set of statements called *axioms* or *postulates* or *laws*. Laws or Postulates are statements that are presented without proof; they cannot be proven; we believe them to be true because they have been *experimentally verified*. (E.g. Newton's 2nd Law, $=\vec{\boldsymbol{F}}_{net} = m\vec{\boldsymbol{a}}$ is a postulate; it cannot be proven from more fundamental relations. We believe it is true because it has been abundantly verified by experiment.)

Newton's 2nd Law also has a limited *regime of validity*. If objects go very fast (approaching the speed of light) or are very small (microscopic, atomic), then this "law" begins to make predictions that conflict with experiment. However, within its regime of validity, classical mechanics is quite correct; it works so well that we can use it to predict the time of a solar eclipse to the nearest second, hundreds of year in advance. We can send a probe to Pluto and have it arrive right on target, right on schedule, 8 years after launch. Classical mechanics is not wrong; it is just incomplete. If you stay within its well-prescribed limits, it is correct.

Each of our theories, *except* relativistic Quantum Mechanics, has a limited regime of validity. As far as we can tell (to date), QM (relativistic version) is *perfectly* correct. It works for *all* situations, no matter how small or how fast. Well... this is not quite true: no one knows how to properly describe gravity using QM, but everyone believes that the basic framework of QM is so robust and correct, that eventually gravity will be successfully folded into QM without requiring a fundamental overhaul of our present understanding of QM. String theory is our current best attempt to combine General Relativity and QM (some people might argue "String Theory" is perhaps not yet really a theory, since it cannot yet make (many) predictions that can be checked experimentally, but we can debate this!)

Roughly speaking, our knowledge can be divided into regimes like so:



In this course, we mostly stick to the upper left quadrant of this figure. But we will show how non-relativistic QM is completely compatible with non-relativistic classical mechanics. (i.e. how QM agrees with classical mechanics, in the limit of macroscopic objects.)

To get some perspective, let's step back, and ask **What is classical mechanics (C.M.)?** It is, most simply put, the study of how things move! Given a force, what is the motion? So, C.M. studies ballistics, pendula, simple harmonic motion, macroscopic charged particles in **E** and **B** fields, spinning tops, etc. You might use the concept of energy (and conservation laws) to make life easier. This leads to new tools beyond just Newton's laws: e.g. the Lagrangian, L, and the Hamiltonian,H, describe systems in terms of different (but still conventional) variables. With these, C.M. becomes more economical, and solving problems is often simpler. (At the possible cost of being more formal) Of course, what one is doing is fundamentally the same as Newton's **F** = m **a**.

The equations of motion are given (in various formalisms) with equations like:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \text{or} \quad \begin{cases} \frac{\partial H}{\partial x} = -p_x \\ \frac{\partial H}{\partial p_x} = x \end{cases}, \quad \text{or} \quad \mathbf{F} = \text{m}\,\mathbf{a}$$

(But if you've forgotten the Lagrangian or Hamiltonian approaches, it's fine for now…)

The general goal of C.M. is to find the equation of motion of objects:
*Given initial conditions, find **x**(t) and **p**(t), position and momentum, as a function of time.*

Then, you can add complications: E.g. allow for more complicated bodies which are not pointlike, ask questions about rotation (introduce the moment of inertia, and angular momentum (**L**=**r**x**p**), move to many-body systems (normal modes), etc…

Q.M. is about the same basic thing: *Given a description of the energetics of a system, what are the observable outcomes of measurements?* Q.M. tends to focus on small systems. (Technically, systems with small *action*, $\int Ldt$ of order $\hbar$) The idea of "measurement", and even what we mean by "motion", will have to be generalized.

Having just completed C.M., your initial reaction may be "but, size doesn't matter"! Neither L nor H cares about size, and C.M. often deals with so called "point objects". (Isn't a point plenty small?!) Unfortunately, in a certain sense, *everything you learned in 2210 and 3210 is WRONG!* To be a little more fair, those techniques are fine, but only applied to real-world sized objects - there's a *regime of validity*. Ultimately, C.M. breaks down: if you try to apply the Lagrangian/Hamiltonian formalism to an electron in a B-field (or in an atom) or an atom in a trap, a quark in a proton, a photon in a laser beam, etc… you will fail big time!

It's not just that the equations are wrong. You can't patch them up with clever correction terms, or slight modifications of the equations, like *relativity* does at high speeds. The whole MIND SET is wrong! You cannot *ask* for ***x(t) and p(t)***. It's not well defined! Point particles *do not exist.* Particles have a wave nature, and waves have a particle nature. There is a duality in the physical world, which is not classical. This is often called *complementarity.* We must start from *scratch*, and develop a whole new framework to describe small systems.

There are many new ideas involved. Some are formal and mathematical, some are rather unintuitive, at least at first. As a colleague of mine once explained, QM is kind of like trying to learn Swahili slang. First, you must learn a new language, and then you must learn a new *culture,* and only then can you finally begin to truly understand the slang... I will try to motivate as much as possible, and we'll study plenty of concrete examples. Quantum mechanics comes from experiment!

The laws (axioms, postulates) of Classical Mechanics are short and sweet: Newton's Laws. (You might add "conservation of energy" if you want to extend C.M. to include thermodynamics.) You can add two more postulates (that the laws of physics are the same in all inertial frames, and the speed of light is constant) to extend C.M. to *include* special relativity. The laws of classical electricity & magnetism (which still falls under an umbrella of Classical Mechanics) are similarly short and sweet: add in Maxwell's (4) equations plus the Lorentz force law.

Alas, there is no agreement on the number, the ordering, or the wording of the Postulates of Quantum Mechanics. Our text makes a pretty standard choice… Quantum Mechanics has (roughly, depends who is counting) 6 Postulates. They cannot be stated briefly; when stated clearly, they are rather long-winded. Compared to Classical Mechanics, quantum mechanics is an unwieldy beast – maybe a bit scary and ugly at first sight, but very, very powerful.

As we go along, I will write the Postulates as clearly as I can, so that you know what is assumed and what is derived. Writing them all down now will do you little good, since we haven't yet developed the necessary vocabulary. But I will try, by writing partially correct, but incomplete or inaccurate versions of each Postulate, just so we can get started. Later on, when ready, we will write the rigorously accurate versions of the postulates.

Don't worry if these seem rather alien and unfamiliar at first - this is really the subject of the entire course - we're just making our first pass, getting a kind of overview of where we're heading. So let's start.

First, some essential language and math ideas, mostly coming from linear algebra.

When we have a physical system, we will represent its physical state in an abstract and compact way. Our notation will be something like this: $|\Psi\rangle$

That letter inside is the Greek, psi, pronounced "sigh". Or we might use a lower case psi, $|\psi\rangle$ We can put any symbol in there, depending on what information we want to share! The part that matters, symbolically, is not the specific "name" inside, but the "|" and ">" surrounding it. Because those kind of look like the right-hand part of a bracket symbol < | > in math, we call this state a "ket". I will freely talk about "the quantum state of an object", or "the ket" as synonyms. This is our way of simply presenting useful information about some system in a shorthand way.

By way of analogy, you have gotten used to similarly abstract abbreviations and notations rather like this before. For instance when you say a classical particle has momentum $\vec{\boldsymbol{p}}$ or position $\vec{\boldsymbol{r}}$, that "arrow on top" means "a vector". This is a very shorthand notation hiding a lot of information (Hidden in there are three components, so there are three real numbers "buried" in that arrow, along with rules for how to manipulate it) Like in the ket, we freely choose to put different symbols under the vector, so using p we mean momentum, or L for angular momentum... When you FIRST saw vector notation, the arrow on top might have seemed weird, and abstract. Now, it just means… a vector! You know how to work with vectors (how to dot them, how to cross them, how to scale them, how to sketch them) and so at this point they are familiar friends. Different vectors get different labels, but always with the arrow on top. So similarly, different states will have different labels inside, but always with the "ket" notation. Kets will be familiar friends very soon too!

When we make measurements, there's going to be an operator associated with that. An operator (I might call it **A**) is a mathematical object – it acts on kets (on quantum states), and yields different kets… You've used operators before, of course - like the operator "d/dx" which acts on a function, takes its derivative, and gives you a different function. In E&M, the "taking the curl" operator acts on an E-field and gives you physical information (it tells you the (-) time rate of change of the B-field, which itself is an "operator" acting on B!) There, the operator "curl", which acts on a vector function, gives you a different vector function. (Another operator, one which acts on spatial vectors, might be e.g. the "rotate clockwise by 90 degrees" operator….) In quantum mechanics we will find operators associated with all measurements (like measuring the z-component of spin, or the position of an object, etc…) Part of our job is to figure out what those operators are. Once you know them, you will already know a lot about the outcome of experiments.

So a quantum operator **A** acts on a ket and gives you a different ket. But sometimes, for certain *special* kets, acting **A** on it gives you the same state back *unchanged*, except for an overall multiplier. Such special states are called *eigenstates*, ("eigenkets") and the overall multiplier is called the *eigenvalue*. Thinking back to simpler analogies, if you "operate" on a vector, you might stretch it, rotate it, flip it, alter it,… turning it into another vector. If your operator acts on a particular vector and comes up with the same vector doubled in length – you would say the vector was fundamentally unchanged except for an overall multiplier of the number 2. So that would be an "eigenvector", with "eigenvalue 2")

If you haven't had linear algebra in a while, you might want to review the chapter on eigenvalues, but not to worry - the *procedure* is straightforward, at least in the electron-spin case, so you'll get plenty of practice here. For the situations we will start with, many operators of interest will have only two possible eigenvalues!

Here is the quantum notation: If $|\psi\rangle$ is one of those special states (an eigenstate) then
$\mathbf{A}\,|\psi\rangle = a\,|\,\psi\,\rangle$,
where the little "a" on the right is just a number (a simple multiplier). The operator **A** acts on a state, and gives us BACK that same state, except with that multiplier "a" out front.
Since this is a special state I should really give it a name that acknowledges its special status. Calling it $|\psi\rangle$ is so generic… we should give it a label that immediately tells us what the eigenvalue is going to be! McIntyre would, in this case, rename this particular special ket $|a\rangle$, so he would say $\mathbf{A}\,|a\rangle = a\,|a\rangle$.
Yes, that's really compact! In words: the first **A** is the operator, it acts on the ket $|a\rangle$, and the result is a new ket, which is the NUMBER "a" multiplied by the ket we started with, $|a\rangle$.

Every "ket" has a partner. Because "ket" was named for the right half of a "bracket < | >", the partner, (the left half of the bracket) is called the "bra" (Don't blame me, this is the notation!) We write it just as you might expect: the partner of the ket $|\psi\rangle$ is the bra $\langle\psi|$.

The main idea here is that kets are a lot like vectors, and the bra is useful because we like to "dot" vectors. The analogy here is going to be that the "dot" of a bra with a ket is a bracket, namely $\langle\psi|\psi\rangle$, or, if bra and ket are different objects, $\langle\phi|\psi\rangle$. We'll get into this linear algebra soon – for now, don't sweat the details! With that mathematical notation as background, let's write down the postulates of QM.

Like I said, this is not going to mean a whole lot to you just yet, but as the term goes by you can (should!) revisit this, and see that it makes more sense, that the words have more concrete meanings for you, that you have examples in mind where you can USE these postulates to calculate and predict experimental outcomes!

Postulate 1: The state of a physical system is *completely* described by a mathematical object, called the quantum state or ket, written with the novel notation (see discussion above): $|\Psi\rangle$

The ket is not "the particle", or "the position of the particle", or "the spin of the particle", it is merely a mathematical object which represents *information about* the particle.

You might compare/contrast this with a classical analogy, e.g. $\vec{\boldsymbol{r}}(\boldsymbol{t})$ is not "the particle", but it is "the position of the particle as a function of time", and contains a lot of useful information about the particle – but not everything (what about mass, charge, spin, etc…)

At first, the ONLY property we will pay any attention to is the "spin" of the electron, specifically the x-, y-, and z- components of its spin. Then we'll talk about the energy of the system. Later, we will add in spatial information (position and momentum), and consider situations more complicated than just "an electron in a B-field" – at which point the math will ramp up a bit in difficulty, but all the fundamental ideas will remain the same!

Postulate 2 says that outcomes of measurements ("physical observables") will always have associated with them a mathematical operator. (See the discussion a few pages earlier about what an operator is and does…) For now, this is just the basic claim that we CAN find a mathematical operator associated with any/all observables.

Postulate 3: The ONLY possible outcomes of a measurement of a physical observable is one of the "eigenvalues" of the operator **A** associated with that measurement.

Postulate 3 does not tell you what the outcome will be, only what possibilities there are. (The outcomes are covered more in postulate four, but are not always determinate!) Since eigenvalues of many operators are discrete (and limited in number) this is where the "quantum" aspect starts to kick in – not all measurement outcomes are observed, only those that are eigenvalues of the measurement operator! That's our first "quantum weirdness" (classically most measurements will yield a continuum of possibility outcomes.)

Given a quantum state $|\psi>$, if you measure a physical observable (with associated operator **A**), we just said (postulate 3) that the only possible outcomes are ONE of the eigenvalues. If the operator A has several possible eigenvalues, we might label them a_n where e.g. a_1 is one possibility, a_2 is another, etc. And now we get to our key postulate:

Postulate 4: the probability of measuring any one particular eigenvalue a_n is given by
$Prob(\text{measuring one specific result } a_n) = |\langle a_n|\psi\rangle|^2$
(Where $|a_n>$ is the normalized eigenvector of the operator **A** with the eigenvalue a_n)
Look again at the previous math overview pages to understand this notation. We'll practice with this a lot – when we start dealing with specific examples it won't look so abstract.

Postulate four is another non-classical result – quantum physics does not and can not tell you exactly what the outcome of any possible experiment is. In many situations, it can only tell you what outcomes are possible, and the *probability* of any given outcome. This is different from our "deterministic" worldview, where we have come to believe that if we know everything about a system, we can precisely predict the state of the system in the future. In QM, there is some indeterminacy – postulate 4 rigorously tells you the "odds", but not the specific outcome in any one given case.

Postulate 5: When you measure something (with associated operator **A**), and you get some given result (a_n, one of the eigenvalues of **A** from postulate 3, with odds given in postulate 4), the system will now be in a *new state.* This postulate gives a formula that tells us what the new state is (but the formula involves something called a "projection operator" P_n that I haven't defined yet. It's coming in Chapter 2) Here it is, for later reference:

$$New\ state\ |\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{|\langle\psi|P_n|\psi\rangle|}}$$

The numerator involves an operator P_n, which in words tells you "the projection of a state into a particular eigenstate we have labeled by a_n". The denominator is a number, needed to make the outcome state normalized. This is likely not familiar language or notation yet (!) The point is - the new state is well defined, *given* some experimental outcome. When you *make* a measurement, even if you can't predict it, after you make it you know your exit state.

Postulate 5 is sometimes called the "collapse" or "measurement" postulate. We have taken a state, made a measurement, got an outcome, and now we have a new/different state than we started with. Measuring quantum systems can affect them!

A lot of people struggle with this postulate, including many great physicists who want to know the mechanism of this collapse. Lots of philosophy of science (and argument) has gone into making sense of this postulate, and there are people who choose different language (and thus interpretation) than this. But it *works*, and it's a postulate – we accept it to see where it leads us!

Postulate 6: As time goes by, quantum states evolve. This "time evolution" (in the absence of measurements!) is given by a simple equation, involving the Hamiltonian (the same one from classical physics, which I think of informally as a formula for the energy of the system) Measurement of energy is associated with an operator, we name that operator **H**, and then time evolution in QM is given by Schrodinger's (time-dependent) equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)\ |\psi(t)\rangle$$

"i" here is the complex number ($\sqrt{-1}$), $\hbar$ is "h-bar" is Planck's constant over 2π, and this is a first order differential equation whose solution tells you the state as a function of time.

One's first reaction to Postulate 6 might be "Where did <u>that</u> come from?" How on earth did Schrödinger think to write that down? We will try to make this equation plausible and show the reasoning that lead Schrödinger to this Nobel-prize-winning formula. But, remember, it's a *Postulate*, so it cannot be derived. We believe it is true because it leads to predictions that are experimentally verified.

At this moment, I realize these postulates look totally formal and abstract and not at all like "physics"… They will become more and more concrete for you as we look at physical examples.

Here's a summary so far of where we are:

The Conventional Umpire:	"I calls 'em as I see 'em."
The Classical Umpire:	"I calls 'em as they <u>are</u>."
The Quantum Umpire:	"They ain't <u>nothing</u> till I calls 'em."

In quantum mechanics, we are not allowed to ask questions like "What is the particle doing?" or "Where is the particle?" or “what is the angular momentum vector for this particle”? Instead, we can only ask about the possible results of specific measurements. For instance, we might ask “if I measure the z-component of the spin of this electron in a particular well-described experimental apparatus, what outcomes might I get, and what is the probability that I will get such-and-such a result??”

QM is all about measurement, which is the only way we know anything about the physical universe. Quantum Mechanics is fundamentally a probabilistic theory. This indeterminacy was deeply disturbing to some of the founders of quantum mechanics. Einstein and Schrödinger were never happy with these postulates. Einstein was particularly unhappy and never accepted QM as complete theory. He agreed that QM always gave correct predictions, but he didn't believe that the ket (the quantum state) contained all the information describing a physical state. He felt that there must be other information ("hidden variables"), which if known, would allow an exact, deterministic computation of the result of any measurement. In the 60's and 70's, well after Einstein's death, it was established that "local hidden variables" theories conflict with experiment (!) Einstein was wrong on this one - the postulates of QM are consistent with experiment! The “ket” really does contain everything there is to know about a physical system, and it only allows probabilistic predictions of the results of measurements.

QM is infuriatingly vague about what exactly constitutes a "measurement". How do you actually measure observable properties of a particle? For a position measurement, you could have the particle hit a fluorescent screen or enter a bubble chamber. For a momentum or energy measurement, it's not always so clear. For the z-component of spin… well that one is pretty straightforward, and was historically first done back in 1922. That’s where we will start the term! But for now, "measurement" is any kind of interaction between the microscopic system observed and some macroscopic (many-atom) system, such as a screen, which provides information about the observed property.

Statistics review.

Because QM is fundamentally probabilistic, let's review some elementary statistics. In particular (to start) let's consider random variables that can assume *discrete* values. Suppose we make many repeated measurements of a random discrete variable called x. An example of x is the mass, rounded to nearest kg (or height, rounded to the nearest cm) of a randomly-chosen adult.

We label the possible results of the measurements with an index i. For instance, for heights of adults, we might have x_1=25 cm, x_2 = 26 cm, etc (no adult is shorter than 25 cm). The list $\{x_1, x_2, \ldots x_i, \ldots\}$ is the called the *spectrum* of possible measurement results. Notice that x_i is **not** the result of the i^{th} trial (the common notation in statistics books). Rather, x_i is the i^{th} possible result of a measurement in the list of all possible results.

N = total # of measurements.
n_i = # times that the result x_i was found among the N measurements.

Note that $N = \sum_i n_i$ where the sum is over the *spectrum* of possible results, *not* over the N different trials.

In the limit of large N (which we will almost always assume), then the *probability* of a particular result x_i is $P_i = \frac{n_i}{N}$ = (fraction of the trials that resulted in x_i).

The average of many repeated measurements of x = **expectation value** of x =

$$\langle x \rangle = \frac{sum\ of\ results\ of\ all\ trials}{number\ of\ trials} = \frac{\sum_i n_i\, x_i}{N} = \sum_i \left(\frac{n_i}{N}\right) x_i = \sum_i P_i\, x_i$$

The average value of x is the weighted sum of all possible values of x: $\langle x \rangle = \sum_i P_i\, x_i$
This is the *expectation value of x* (even though you might e.g. NEVER find any particular individual whose height is the average or "expected" height!)
We can generalize this result to any function of x:
$\langle x^2 \rangle = \sum_i P_i {x_i}^2$, or $\langle f(x) \rangle = \sum_i P_i f(x_i)$
The brackets $\langle \ldots \rangle$ means "average over many trials". We would call this the "expectation value of x^2".
A measure of the expected spread in measurements of x is the standard deviation σ, defined as "the rms average of the deviation from the mean".
"rms" = root-mean-square = take the square, average that, then square-root that.
$\sigma = \sqrt{\sigma^2} = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ (σ^2 is called the *variance*.)

Let us disassemble and reassemble: The *deviation* from the mean of any particular result x is $\Delta x = x - \langle x \rangle$. The deviation from the mean is just as likely to be positive as negative, so if we average the deviation from the mean, we get zero: $\langle \Delta x \rangle = \langle x - \langle x \rangle \rangle = 0$.

To get the average or typical *size* of Δx, we will square it first, before taking the average, and then later, square-root it: $\sigma = \sqrt{\langle \Delta x^2 \rangle} = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$

It is not hard to show that another way to write this is $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$:
There are times when this way of finding the variance is more *convenient*, but the two definitions are mathematically equivalent:

Proof:

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle = \sum_i P_i \cdot (x_i - \langle x \rangle)^2 = \sum_i P_i \cdot (x_i^2 - 2x_i\langle x \rangle + \langle x \rangle^2)^2$$

$$= \sum_i P_i x_i^2 - 2\langle x \rangle \sum_i P_i x_i + \langle x \rangle^2 \sum_i P_i = \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2$$

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

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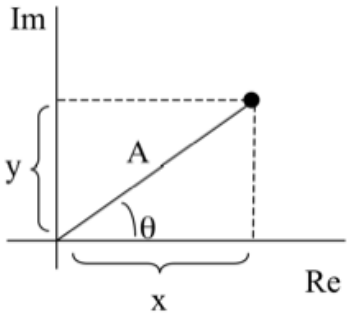
Complex Number Review: Quantum states are in general complex objects. So it's worth a quick review of complex numbers, since we'll be dealing with this all term.

$i = \sqrt{-1},\ \ i \cdot i = -1,\ \ so\ i = -\frac{1}{i}\ or \frac{1}{i} = -i$

Any complex number *z* can always be written in either

Cartesian form: z = x+iy *or* Polar form: $z = Ae^{i\theta}$



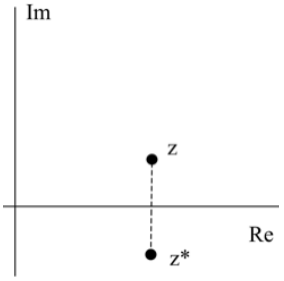
You can visualize a complex number by thinking of it as a as a point in the complex plane:

This picture also matches up with one of the most important theorems of complex numbers, *Euler's relation:*

$\boxed{e^{i\theta} = cos\theta + isin\theta}$ (Which can be proven w/ a Taylor Series expansion). It says:

Re[z] = x = A cosθ

Im[z] = y = A sinθ (Look at the picture above, do you see the connections?)



The *complex conjugate* of *z* is called $z^* = Ae^{-i\theta}$. Note that:

$z \cdot z^* = (x + iy)(x - iy) = x^2 + iyx - ixy + y^2 = x^2 + y^2$

is purely real.

We call |*z*| = "modulus of *z*" or "amplitude of *z*", and define it as

$|z| = \sqrt{x^2 + y^2}$, and from the above, $|z|^2 = z \cdot z^*$.

Notice that $z^2 = z \cdot z \neq z \cdot z^* = |z|^2$ (Squaring complex numbers does NOT always yield a real result, and in general is different than multiplying by the complex conjugate. The square of a complex number is DIFFERENT from the square of the amplitude of that number.)

Here's a useful fact:

$e^{z_1 \cdot z_2} = e^{z_1} \cdot e^{z_2}$ ___(where *z_1, z_2 are any 2 complex numbers)*

This means in particular that $e^{(\alpha+\beta)} = e^{\alpha} \cdot e^{\beta}$

(which in turn can be used to derive various trig identities, like e.g. that cos(a+b) = cos(a)cos(b)-sin(a)sin(b): just look at the real part of the equation)

Also, if $z_1 = A_1 e^{i\theta_1},\ z_2 = A_2 e^{i\theta_2}$ then it is quick to find the product:

$z_1\, z_2 = A_1 A_2 e^{(\theta_1+\theta_2)}$

<u>One more useful fact about complex numbers:</u>

Any complex number z, written as a complicated expression, no matter how messy, can be turned into its complex conjugate *z** by replacing every i with -i, so e.g.

$z = \frac{(5+6i)(-7i)}{(2i+3e^{-i\theta})}\ \ means\ \ z^* = \frac{(5-6i)(7i)}{(-2i+3e^{i\theta})}$

