5.22.

Ampere's Law: + "Gauss for Magnets"

These are two of Maxwell's eq'ns, tells you the connection between current + B field. It's the magnetic analogy of

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \quad \longleftrightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \quad \longleftrightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \text{(Ampere's Law)}$$

(All of these are for electro/magneto statics, + will be fleshed out later with time dependence!)

Could say "these are exp'tal facts" + go from there!

But it's nice to see consistency/connection back to Biot-Savart

I claim: From Biot-Savart $\longrightarrow$ you can show Ampere's Law *

" Ampere's Law $\longrightarrow$ " " " Biot-Savart **

Much like From Coulomb $\rightarrow$ you can show Gauss' law

From Gauss' Law $\rightarrow$ " " " Coulomb's

+ Like there, the "Maxwell Form" turns out to be deeper + broader.

" " , both may prove useful in different circumstances!

---

* is hard - see Griff 5.3.2

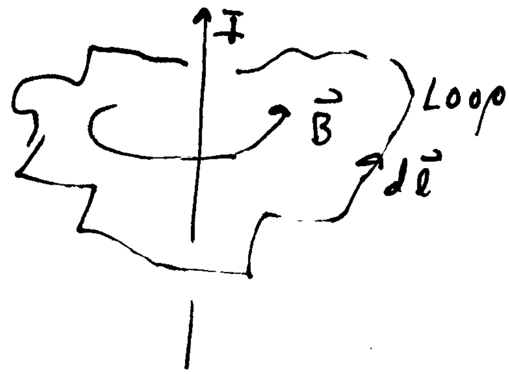
** is harder!

5.23

Plausibility argument:

we know of <u>one</u> case (∞ wire) we can solve for $\vec{B}$ with Biot-Savart,

we <u>did</u> this + got $\vec{B}(\vec{r}) = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$

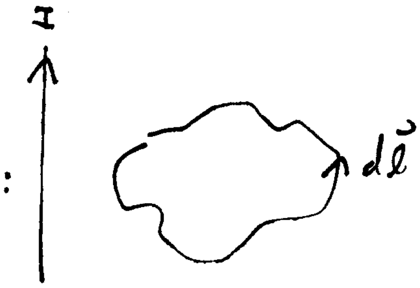


In this case, $\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\underbrace{\hat{\varphi}\cdot d\vec{l}}_{s\,d\varphi}$

<u>Any</u> loop of any shape <u>around</u> the wire

$= \frac{\mu_0 I}{2\pi}\cdot\oint d\varphi = \mu_0 I.$

If the loop had not gone <u>around</u> I, like this:



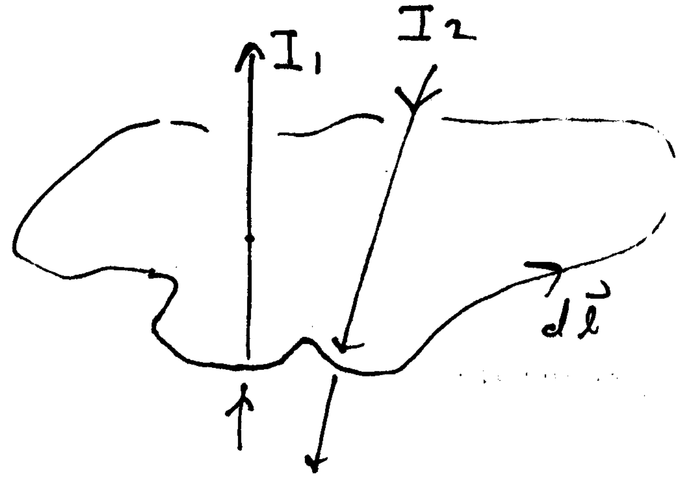
then $\int d\varphi = 0$!

so here $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$

By superposition, with multiple currents, $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{total poking thru loop}}$.

- Note: If loop is opposite direction ($d\vec{l}$ goes other way) $\Rightarrow$ − sign, which means I is + if it "pokes" through loop in RH sense
  I is − " " " " " " LH sense

So e.g



$\oint \vec{B}\cdot d\vec{l} = \mu_0(|I_1| - |I_2|)$ here.

Also, $I_{enc} = \underbrace{\iint}_{} \vec{J} \cdot dA$ ← Just def of $\vec{J}$!

where this covers the loop area *

so $\underbrace{\oint \vec{B} \cdot d\vec{l}}_{} = \mu_0 \iint \vec{J} \cdot d\vec{A}$

Stokes

$\iint (\vec{\nabla} \times \vec{B}) \cdot d\vec{A} = \mu_0 \iint \vec{J} \cdot d\vec{A}$ true for any / every loop

$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$ which is Ampere's Law.

This is not a proof, it assumed ∞ wires ... but it shows consistency.

Proof involves starting with Biot-Savart: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{\mathscr{R}}}{\mathscr{R}^2} d\tau'$

then formally showing $\vec{\nabla} \cdot \vec{B} = 0$

and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r})$ By taking $\vec{\nabla} \cdot$ or $\vec{\nabla} \times$ this

+ "massaging". It's a lovely exercise in vector calc.

---

* Puzzle: "$I_{through}$" $= \iint \vec{J} \cdot d\vec{A}$. But there are many surfaces sharing the same boundary loop. Which do you use?

(A: It's $I_{through}$! Any/All ok)

5.25

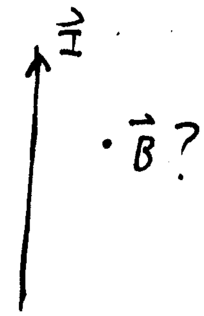
Applying Ampere's Law: For steady currents

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \qquad \text{or} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

any loop — I "poking through" that loop, (in the R.H. sense)

Like Gauss, Ampere's law let's us find $\vec{B}$ if we have symmetry + can argue up front that B is constant +/or can be pulled out of the integral. (Just like Gauss)

Ex #1: The Infinite wire:



- Claim: $\vec{B}$ cannot be radial, this would violate $\vec{\nabla}\cdot\vec{B} = 0$ or $\iint \vec{B}\cdot d\vec{A} = 0$

  (consider a Gaussian "can" around the wire)

- Claim: $\vec{B}$ cannot have any ~~B_z~~ z dependence, or φ dependence (symmetry!)
- Claim: $\vec{B}$ " " " $\hat{z}$ component (why?) ← (Biot-Savart!)
- So $\vec{B} = B(s)\,\hat{\varphi}$, just by symmetry.

then $\oint \vec{B}\cdot d\vec{l} = B\oint dl = B\cdot 2\pi s = \mu_0 I$

pick a circular loop

so $B = \dfrac{\mu_0 I}{2\pi s}$ (which we knew from a horrible integration of Biot-Savart, this is much easier!)

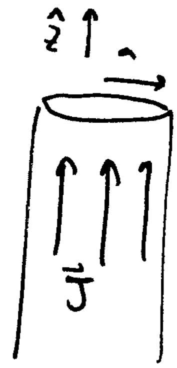
5-26

Example II: A thick wire

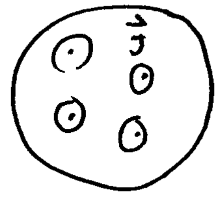
radius a,

uniform $\vec{J} = J_0 \hat{z}$ for $s \leq a$

$= 0$ for $s > a$



Side



Top view

Draw Amperian Loop, circle centered around origin.

Claim: Like thin wire, $\vec{B} = B(s)\hat{\varphi}$ still. We haven't introduced any z dependence by thickening wire! This is the crucial part, once we know this,

$$\oint_{\text{around Loop}} \vec{B}\cdot d\vec{l} = \mu_0 I_{\substack{\text{thru}\\ \text{loop}}}$$ both sides are simple

L.H.S $\oint \vec{B}\cdot d\vec{l} = \oint B(s)\,dl = B(s)\cdot 2\pi s$

RHS. $I_{\text{thru loop}} = \iint \vec{J}\cdot d\vec{A}$. If $s \leq a$, this gives

$$\iint J_0\, dA = J_0 \cdot \pi s^2$$

If $s > a$, this gives

$$\iint_{\text{out to } s=a} J_0\, da = J_0 \cdot \pi a^2 \left(= I_{\text{tot}}\right)$$

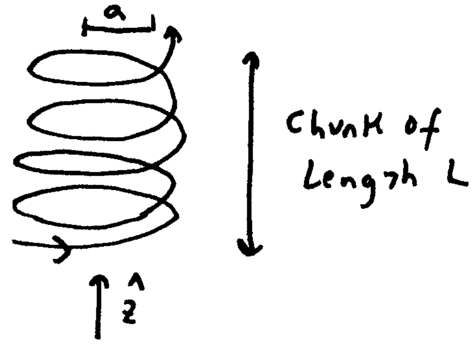
so $s \leq a \Rightarrow \vec{B}(s) = \mu_0 J_0 \dfrac{s}{2}\hat{\varphi}$

$s > a \Rightarrow \vec{B}(s) = \mu_0 \dfrac{J_0 a^2}{2s}\hat{\varphi}$

Note: No surface current $\Rightarrow \vec{B}$ is continuous ✓

$\vec{B}(0) = 0$, makes sense by symmetry!

Example III: Infinite Solenoid



- Radius a
- assume n windings / meter.

(so $nL = 4$ in my little sketch!)

Outside: $\vec{B}$ better not have $\hat{s}$ component (in or out!) → that would make $\oiint \vec{B} \cdot d\vec{A}$ nonzero, violating $\vec{\nabla} \cdot \vec{B} = 0$.

How about a $\hat{\varphi}$ component? Tempting! (what a wire gave)

But no, draw a loop, of radius s

$$\underbrace{\oint \vec{B} \cdot d\vec{l}}_{B_\varphi(s) \cdot 2\pi s} = \mu_0 I_{\text{through}}$$

$$B_\varphi(s) \cdot 2\pi s = \mu_0 \cdot 0$$!! convince yourself!

current is not "poking through", it circles around! well....

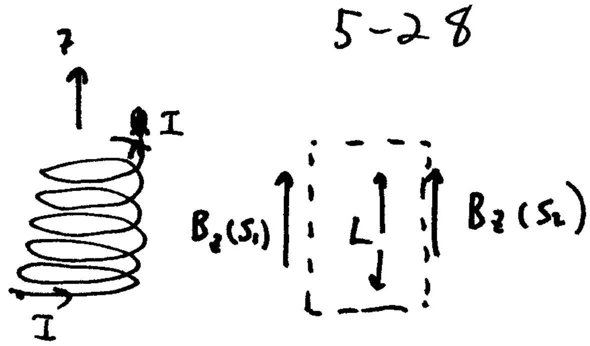
(N.B. I'm assuming we really have $\vec{K}$ ⇛ Purely in $\hat{\varphi}$ direction

In reality, I does flow up page ⇒ Through this loop, so there is the usual $\mu_0 I$ on RHS. But you'll see, this is small + not so important compared to the huge B inside we're going to get)

So maybe $\vec{B} = B_z(s)\,\hat{z}$. Really all we could have.

Claim: $B_z(\infty) = 0$.

- Experimental fact
- could convince yourself from Biot-Savart
- B must vanish far from line currents...



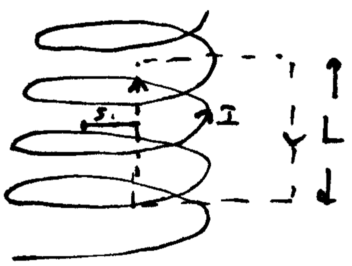
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{thru}$$

$$(B(s_1) - B(s_2))L = 0$$

so $B(s_1) = B(s_2)$

But $B(\infty) = 0$, so $B = 0$ everywhere outside!!

Surprising, important result!



Now use this loop

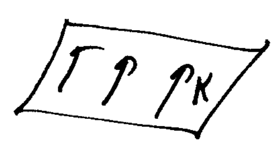
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{through}$$

$$B(s_1)\cdot L + 0 = \mu_0 \cdot \underbrace{I}_{\text{this current through each wind}} \cdot \underbrace{nL}_{\text{this many windings}}$$

so $\vec{B} = \mu_0 n I \hat{z}$ if $s < a$

$= 0$ if $s > a$.

Sort of "Capacitor-Like": uniform B inside
0 B outside.

See Griff for more examples:  Sheet

Sheet +
Torus are both good ones!

5-29 Griff p.23:

Vector Potential : $\vec{\nabla}\times(\vec{\nabla} f)=0$, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V})=0$

Recall that $\vec{\nabla}\times\vec{E}=0$ says $\vec{E}=-\vec{\nabla}V$

Similarly, $\vec{\nabla}\cdot\vec{B}=0$ makes you want to look for a function $\vec{A}$ such that $\vec{B}=\vec{\nabla}\times\vec{A}$

This would be like a "potential" for $\vec{B}$ ... but also very different.

[Biot-Savart gives us a formula for $\vec{A}$ (homework problem!)]* (soon)

V was nice because · Scalar (simple function)

· $\nabla^2 V = -\rho/\epsilon_0$ or, often, $\nabla^2 V = 0$

$\Rightarrow$ V is often easy to figure out !

e.g $V = \frac{1}{4\pi\epsilon_0}\iiint \rho \frac{d\tau}{\mathscr{R}}$ (· can always add any constant to V, it had that intrinsic ambiguity, though)

or Method of images

or separation of variables

And then $\vec{E}=-\vec{\nabla}V$ is quick ... $\boxed{\text{And}}$ $V = \frac{\text{Energy}}{\text{charge}}$! ← Nice physical interpretation!

Want something like this for $\vec{B}$!

$\vec{A}$ = "vector potential" is not quite as nice, on all counts

- vector, not scalar
- Not such a nice physical "meaning". Still, has many uses...

More about $\vec{A}$:

If $\vec{B} \equiv \vec{\nabla} \times \vec{A}$, and $\underbrace{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$ Maxwell (Ampere) in M-statics

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{A})} = \mu_0 \vec{J}$$

From p.23

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

- Now, remember Voltage V was <u>defined</u> by $\vec{\nabla} \times (-\vec{\nabla} V) = 0$ and we could always add <u>any</u> function, φ, whose gradient vanishes (i.e. a constant!) and <u>still</u> $\vec{\nabla} \times (-\vec{\nabla}(V + \varphi)) = 0$

This time, vector potential $\vec{A}$ is <u>defined</u> by $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

so here we can always add <u>any</u> function, $\vec{\varphi}$, whose <u>curl</u> vanishes + <u>still</u> $\vec{\nabla} \cdot (\vec{\nabla} \times (\vec{A} + \vec{\varphi})) = 0$.

This is <u>more</u> freedom (<u>many</u> functions, interesting ones, have $\vec{\nabla} \times \vec{\varphi} = 0$)

With V, we often picked our "offset" to ensure $V(\infty) = 0$.

This was an extra condition, a <u>choice</u>, to simplify

Let's do same thing here, <u>choose</u> this $\vec{\varphi}$ "freedom" to modify $\vec{A}$ so it satisfies an extra restriction

Popular choice (but there are others!)

Let's make sure $\vec{\nabla}\cdot\vec{A} = 0$ too!

$\longrightarrow$ Must prove we can always do this * (see bottom!)

$\longrightarrow$ If we do this, $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

simplifies to $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$

[* this "separation" valid only for Cartesian coords, though!]

this is really 3 eq'ns*: $\nabla^2 A_x = -\mu_0 J_x$

$\nabla^2 A_y = -\mu_0 J_y$

etc.

each one is "Poisson's eq'n", and we know the sol'n

If $\nabla^2 V = -\rho/\varepsilon_0$, then $V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{R}$

so $\nabla^2 A_x = -\mu_0 J_x \Rightarrow A_x = \frac{1}{4\pi}\mu_0 \int \frac{J_x(r')\,d\tau'}{R}$

This is our sol'n for $\vec{A}$, $\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J}(r')\,d\tau'/R}$

At least, if $\vec{J}$ is "local", like current loops!

(see top of page)
* Proof: Suppose $\vec{\nabla}\times\vec{A}_0 = 0$ but $\vec{\nabla}\cdot\vec{A}_0$ is not zero. Say $\vec{\nabla}\cdot\vec{A}_0 = f(\vec{r})$

then I claim we can find a function $\vec{\varphi}(\vec{r})$ such that

$\vec{\nabla}\times(\vec{A}_0+\vec{\varphi}) = 0$, and $\vec{\nabla}\cdot(\vec{A}_0+\vec{\varphi}) = 0$

Proof continued: what would $\vec{\varphi}$ have to be?

we'd need $\vec{\nabla}\times\vec{\varphi}=0$ to preserve $\vec{\nabla}\times(\vec{A}_0+\vec{\varphi})=0$

" " $\vec{\nabla}\cdot\vec{\varphi}=-f$ to ensure $\vec{\nabla}\cdot(\vec{A}_0+\vec{\varphi})=0$.

I can guarantee $\vec{\nabla}\times\vec{\varphi}=0$ if $\vec{\varphi}\equiv\nabla\lambda$ for some function λ.

So can I find a $\lambda(\vec{r})$ such that $\vec{\nabla}\cdot(\nabla\lambda)=-f$?

well, this says $\nabla^2\lambda=-f$. That's Laplace's eq'n!

And for any f, you can always solve " ".

So for any f, λ exists, and then $\vec{A}_0+\vec{\nabla}\lambda$ gives you $\vec{A}$ with properties you want. (In fact, there are many λ's, 'cause you could always add a constant to it!)

This is all rather formal, but here's bottom line:

Given currents ($\vec{J}$), if you want $\vec{B}$ you have a choice

1) Solve Ampere: $\oint\vec{B}\cdot d\vec{l}=\mu_0\iint\vec{J}\cdot d\vec{A}$ ← only helpful if lots of symmetry

2) Solve Biot-Savart, $\vec{B}=\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}(r')\times\hat{\mathscr{R}}}{\mathscr{R}^2}\,d\tau'$

or, the new way

3) Find $\vec{A}=\frac{\mu_0}{4\pi}\iiint\frac{\vec{J}(r')}{\mathscr{R}}\,d\tau'$ ← (Sometimes this is easier. No cross product, for instance!)

and then $\vec{B}=\vec{\nabla}\times\vec{A}$ is quick + easy. (only good if $\vec{J}$ is local, though) or

4) In general, find (Guess!?) a function $\vec{A}$ such that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \qquad \text{(and, if you like, } \vec{\nabla}\cdot\vec{A} = 0\text{)}$$

(and then, again, $\vec{B} = \vec{\nabla} \times \vec{A}$ is quick + easy)

---

why use vector potential? → It does have a physical interpretation (but we won't investigate this)

→ Sometimes easy to solve for! If so... use it!

→ In quantum mechanics, atom ⟺ light interactions are described not with $\vec{E}$'s + $\vec{B}$'s, but V and $\vec{A}$. So, in a sense, it's very deep, physical, + important!

$\boxed{\text{ASIDE}}$ → $\vec{A}$ is ambiguous (just like adding any constant to V changes nothing, so adding any gradient to $\vec{A}$ changes nothing!)

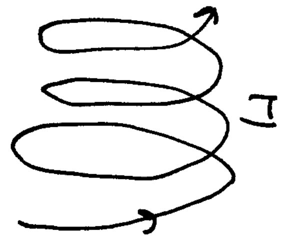
This "freedom" is called "gauge freedom".

We pick $\vec{\nabla}\cdot\vec{A} = 0$ which "picks a gauge" (and this choice is called "the coulomb gauge".) There are other ~~[illegible]~~ choices, some involving time-dependence, might be handier in other situations)

(Gauge freedom turns out to be a central guiding idea in QED, and is ultimately related to conservation of charge + the existence of photons!)

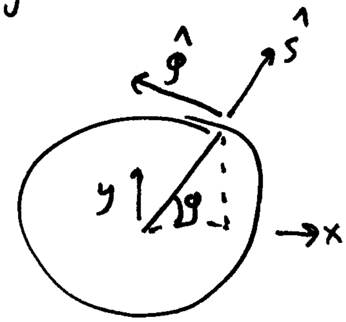
5-34 -alt

Example I: a solenoid Current I
radius R
n turns/meter



we found $\vec{B}$ w. Ampere's Law, but let's find $\vec{A}$.

we want $\nabla^2 \vec{A} = -\mu_0 \vec{J} = -\mu_0 \, n I \, \delta(s-R) \, \hat{\varphi}$ convince yourself!



this $\vec{J}$ is a surface current. On the cylinder

we have $\hat{\varphi} = (-\sin\varphi, \cos\varphi, 0)$ convince yourself!

So $\nabla^2 A_x = +\mu_0 n I \sin\varphi \; \delta(s-R)$

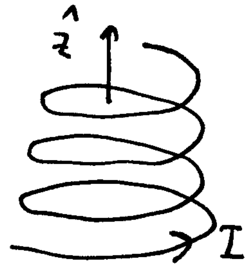
this is the same eq'n you'd get in electrostatics

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ if ρ was $\propto \sin\varphi \, \delta(s-R)$, i.e. a surface charge on a cylinder.

We never solved for V in cylindrical coords, but you could by separation of variables! This sol'n for V_{in} (and V_{out}) would then immediately yield A_x (+ similarly, with $\cos\varphi$, A_y)

But, let's not go this route..

Example I: ∞ solenoid current I, radius R, n turns/meter



We found $\vec{B}$ with Ampere's Law, but let's [redacted] find $\vec{A}$ [redacted].

we want $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ $\left(= -\mu_0 \underbrace{n I \, \delta(s-R)\, \hat{\varphi}} \right.$ but never mind$\left. \right)$

Consider a totally different problem, I solved on p. 26 [redacted]

If $\vec{J} = J_0 \hat{z}$ for $s \le a$ (fat wire)
$\quad\;\; 0$ for $s > a$

we found $\vec{B}(s) = \mu_0 J_0 \dfrac{s}{2} \hat{\varphi}$ $\quad s \le a$

$\qquad\qquad\;\; \dfrac{\mu_0 J_0 a^2}{2s} \hat{\varphi}$ $\quad s > a$

solved $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$.

here, we know

$\vec{B} = B_0 \hat{z}$ for $s \le a$ solenoid. (with $B_0 = \mu_0 n I$)
$\quad\;\; 0$ " $s > a$

and $\vec{\nabla} \times \vec{A} = \vec{B}$. Do you see this is <u>same</u> math problem?

By <u>inspection</u>

$$\vec{A} = \frac{B_0 s}{2} \hat{\varphi} \qquad s \le a$$

$$= \frac{B_0 a^2}{2s} \hat{\varphi} \qquad s > a$$

Note: $\vec{A}$ "looks" a bit like <u>current</u>, in that it's azimuthal. At least in direction check: $\nabla \times \vec{A} =$ From flyleaf: $-\frac{\partial A_\phi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\hat{z}$

inside $\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{B_0 s^2}{2}\right) = B_0\hat{z}$

outside $\frac{1}{s}\frac{\partial}{\partial s}\left(\frac{B_0 a^2}{2}\right) = 0.$

Check is out. $\vec{B} = 0$ out there, but $\vec{A}$ is not. This has real Quantum consequences!

yay! $\vec{\nabla} \times \vec{A} = \vec{B}$, as desired

check $\vec{\nabla} \cdot \vec{A} =$ from flyleaf: $\frac{1}{s}\frac{\partial A_\phi}{\partial \phi} = 0$ ✓.

yay! $\vec{\nabla} \cdot \vec{A} = 0$, as desired.

Also Notice $\vec{\nabla} \times \vec{A} = \vec{B}$

so $\oint \vec{A} \cdot d\vec{l} \underset{\text{Stokes!}}{=} \iint (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \iint \vec{B} \cdot d\vec{a} = \underbrace{\Phi_B}_{\text{Definition of magnetic flux}}$

So here's one "interpretation" of $\vec{A}$: the circulation of $\vec{A}$ tells you magnetic flux through a loop.

In this case, inside $\oint \vec{A} \cdot d\vec{l} = \frac{B_0 s}{2} \cdot 2\pi s = \pi s^2 B_0$ ✓ $= \Phi_B$

outside $\oint \vec{A} \cdot d\vec{l} = \frac{B_0 a^2}{2s} \cdot 2\pi s = \pi a^2 B_0$ ✓ $= \Phi_B$

But... we used $\vec{B}$ to find $\vec{A}$. That's a little silly, why bother finding $\vec{A}$ if we already know $\vec{B}$? So, another ex... well, just trying to get comfy visualizing $\vec{A}$ for now!

Example I': $\vec{B} = B_0 \hat{z}$ everywhere. (Uniform $\vec{B}$)

want $\vec{\nabla} \times \vec{A} = \vec{B} \Rightarrow \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} = 0$ $(B_x = 0)$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0 \qquad (B_y = 0)$$

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0 \qquad (B_z = B_0)$$

Simple enough... e.g. $A_x = 0$, $A_z = 0$, $A_y = B_0 x$ works.

oh, but so does $A_x = -y B_0$, $A_y = 0$, $A_z = 0$!

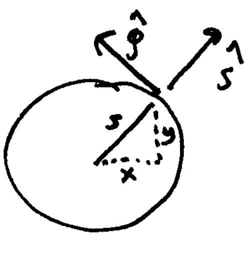
and so does linear combo $A_x = -y \dfrac{B_0}{2}$, $A_y = \dfrac{B_0 x}{2}$, $A_z = 0$.

All 3 of these also satisfy $\vec{\nabla} \cdot \vec{A} = 0$, by the way!

Note $\oint \vec{A} \cdot d\vec{l} = \Phi_B$ says $\underbrace{\oint \vec{A} \cdot d\vec{l}}_{\text{loop of radius } a} = B_0 \pi a^2$

$\underline{\text{If}}$ $\vec{A} = \dfrac{B_0 s}{2} \hat{\varphi}$, then this is true .... so that's also a sol'n.

and $\vec{\nabla} \times \vec{A} = B_0 \hat{z}$ from front flyleaf.



Note: $\hat{\varphi} = \left(-\dfrac{y}{s}, +\dfrac{x}{s}, 0\right)$

so $A_x = B_0\left(-\dfrac{y}{2}, \dfrac{x}{2}, 0\right)$ This solution! is same as we had above

Once again – loss of "freedom" for $\vec{A}$. (This is just like a Big solenoid)

Example II: Let's do a problem where we don't *use* $\vec{B}$ to get $\vec{A}$!

(But, pick one we do know *answer* to, so can check!)

∞ long wire, radius a,
current $I_0 \hat{z}$.

So $\vec{J} = I/\pi a^2 \ \hat{z}$ *inside* wire. $(J_x = J_y = 0)$

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x \, d\tau'}{\mathscr{R}} = 0, \quad \text{same for } A_y.$$

For A_z, we have $\boxed{\nabla^2 A_z = -\mu_0 J_z}$ (constant out to a, uniform in z) I could do the integral,

but I'd rather not. We've *seen* this equation,

think of long wire with uniform λ (or ρ)

$$\nabla^2 V = -\rho/\varepsilon_0 \quad \text{(constant out to } a\text{, uniform in } z\text{.)}$$

I know sol'n to this! Gauss' law gives $\vec{E} = \dfrac{Q_{enc}/\varepsilon_0}{2\pi s L}$

Let's just stay *outside*:

$$= \frac{\lambda \cancel{L}}{2\pi\varepsilon_0 s \cancel{L}}$$

and $V = -\int \vec{E}\cdot d\vec{l}$

$$= \frac{-\lambda}{2\pi\varepsilon_0}\ln s = -\frac{\rho\cdot\pi a^2}{2\pi\varepsilon_0}\ln s$$

so by inspection

$$A_z = -\frac{\mu_0 a^2 J_z}{2}\ln s = -\frac{\mu_0 I}{2\pi}\ln s$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \underset{\text{cylindrical}}{\overset{\text{front flyleaf}}{=}} = -\frac{\partial A_z}{\partial s}\hat{\varphi} = +\frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

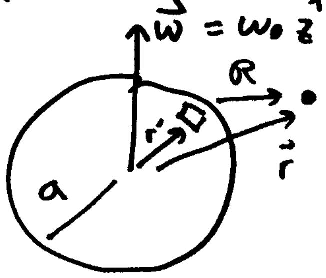
Ah! As it should be!

Ex III: Griffiths solves "rotating sphere of charge".

Sort of a "spherical solenoid".

you found $\vec{K}$ for homework, just need $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\iint \frac{\vec{K}(r')\,da'}{\mathscr{R}}$

Little messy! (Griffiths works it out) I'll do it a little differently:



$\vec{K} = \sigma\vec{v} = \sigma\,\underline{\vec{\omega}\times\vec{r}'}$

remember $\vec{v} = \vec{\omega}\times\vec{r}'$?

so we want $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\sigma \iint_{\text{sphere}} \frac{(\vec{\omega}\times\vec{r}')\cdot \overbrace{a^2 \sin\theta'\,d\theta'\,d\varphi'}^{\text{this is } da' \text{ of patch}}}{\mathscr{R}}$

$\vec{A}(\vec{r}) = \frac{\mu_0 \sigma a^2}{4\pi}\,\vec{\omega}\times\vec{f}(\vec{r})$ with $\vec{f}(\vec{r}) = \iint_{\text{sphere}} \frac{\vec{r}'\sin\theta'\,d\theta'\,d\varphi'}{\mathscr{R}}$

Trick: $\vec{f}$ depends on $\vec{r}$. There is no other vector/preferred direction in this integral, so $\vec{f}(\vec{r}) = C\vec{r}$, there's no other direction for it to point!? C can itself depend on $\vec{r}$, of course.

I let Griffiths do the nasty business of finding C

5-38

Actually, I think we can do it like this:

Suppose we're outside. Let me choose $\vec{r} = (0,0,z)$

then $\vec{f}(\vec{r}) = C\,(0,0,z) = Cz\,\hat{z}$

so $f_z = Cz$. But $f_z = \iint \dfrac{z' \sin\theta' \, d\theta' \, d\phi'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$

$$= \iint \frac{(a\cos\theta')\sin\theta' \, d\theta' \, d\phi'}{\sqrt{a^2 + z^2 - 2zz'}} = a \cdot 2\pi \int_{\theta'=0}^{\pi} \frac{\cos\theta' \sin\theta' \, d\theta'}{\sqrt{a^2 + z^2 - 2za\cos\theta'}}$$

Let $u = \cos\theta'$

$$= 2\pi a \int_{-1}^{1} \frac{u\,du}{\sqrt{a^2 + z^2 - 2azu}} \overset{\text{MMA!}}{=} 2\pi a \cdot \frac{2}{3} \cdot \frac{a}{z^2}$$

Giving $Cz = \frac{4}{3}\pi a^2/z^2$. In this case, we chose $\vec{r} = (0,0,z)$,
so $z = r$ here, i.e.

$$\vec{f}(\vec{r}) = \frac{4}{3}\frac{\pi a^2}{r^3}\vec{r} \quad \text{for } r \text{ outside.}$$

so $\vec{A}(\vec{r}) = \dfrac{\mu_0 \sigma a^4}{3 r^3}\, \vec{\omega} \times \vec{r}$ for $\vec{r}$ outside the sphere.

Inside, the integral I had gives $2\pi a \cdot \frac{2}{3} \cdot \frac{z}{a^2}$

or $Cz = \dfrac{4\pi}{3a} z$, or $C = 4\pi/3a$, thus $\vec{f}(\vec{r}) = \dfrac{4\pi}{3a}\vec{r}$

and $\vec{A}(\vec{r}) = \dfrac{\mu_0 \sigma a}{3}\, \vec{\omega} \times \vec{r}$ for $\vec{r}$ inside

Bottom line $\vec{A}(\vec{r}) = \tilde{C}\ \vec{\omega} \times \vec{r}$

with $\tilde{c}$ given on prev page, (differs for $r >$ or $< a$)

and $\vec{\omega} \times \vec{r} = \omega r \sin\varphi\ \hat{\varphi}$ ← convince yourself.

To get $\vec{B} = \vec{\nabla} \times \vec{A}$, just take curl (front flyleaf)

$$= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\, A_\varphi)\, \hat{r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\varphi)\, \hat{\theta}.$$

or, switching to cylindrical, (noting no z dependence inside), $\vec{A}$ is pure $\hat{z}$! inside

It's a "magnetic dipole", which we'll investigate more, next!

Boundary Conditions

Like $\vec{E}$ fields (+ potential), $\vec{B}$ also has simple behaviours at surfaces / Boundaries.

- $\vec{\nabla} \cdot \vec{B} = 0$ tells us $B^{\perp}_{above} - B^{\perp}_{below} = 0$.

- $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{thru}$ tells us $B_{\parallel}$ is continuous unless you have a surface current $\vec{K}$ at the boundary!

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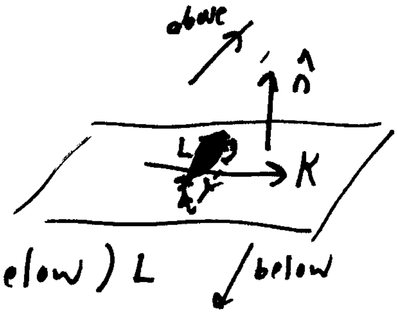
If you <u>do</u> have $\vec{K}$



(Pick loop perp to $\vec{K}$)

$\oint \vec{B}\cdot d\vec{l} = (B_{\parallel}\text{ above} - B_{\parallel}\text{ below})\,L$

But: $\mu_0 I_{\text{through}} = -\mu_0 \cdot (K\,L)$ (see arrows in fig, do you <u>see</u> - sign?)

so $B^{\parallel}_{above} - B^{\parallel}_{below} = -\mu_0 K$

But careful! I picked my loop $\perp$ to $\vec{K}$! So "parallel" here also means parallel to <u>sheet</u>, but $\perp$ to $\vec{K}$!!

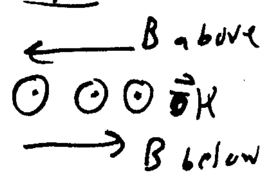
Note that $\hat{n}\times\vec{K}$ points perp to $\hat{n}$, I.E. parallel to sheet
" " $\vec{K}$,

so $B^{\parallel}$ really means $\vec{B}\cdot(\widehat{\hat{n}\times\vec{K}})$ $= \hat{n}\times\hat{K}$

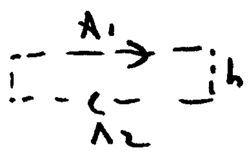
so $\vec{B}_{above} - \vec{B}_{below} = -\mu_0 K$ in the $(\hat{n}\times\hat{K})$ direction,

$= -\mu_0 K\,(\hat{n}\times\hat{K}) = +\mu_0 K\,(\hat{K}\times\hat{n})$

<u>Bit</u> ugly, but idea is simple: $\vec{B}$ is continuous, <u>except</u> that surface currents change $B^{\parallel}$, think e.g.:



For $\vec{A}$, think of $\oint \vec{A}\cdot d\vec{l} = \Phi_{MAG}$. So if Φ is <u>finite</u>



$A_1 L - A_2 L \to 0$ if $h\to 0$. Any loop, any orientation

$\vec{A}$ is <u>always</u> continuous (just like V!)

Summary: See Griffiths triangle on p. 240!

$$\begin{tikzcd} & \rho \arrow[dl, dash] \arrow[dr, dash] & \\ V \arrow[rr, dash] & & E \end{tikzcd}$$

very similar to

$$\begin{tikzcd} & \vec{J} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \vec{A} \arrow[rr, leftrightarrow] & & \vec{B} \end{tikzcd}$$

Given $\vec{J}$ can find $\vec{B}$ (Biot Savart)

" $\vec{B} \Rightarrow \vec{J}$ (Ampere, $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$)

Given $\vec{A}$ can find $\vec{B}$ (easy! $\vec{\nabla}\times\vec{A} = \vec{B}$)

" $\vec{B}$ " can $\vec{A}$ (? Haven't really done this, except in 1-2 special cases)

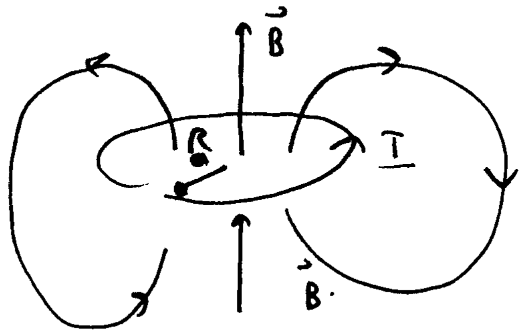
Given $\vec{A}$ can find $\vec{J}$ (easy! $\nabla^2 \vec{A} = -\mu_0 \vec{J}$)

Given $\vec{J}$ " " $\vec{A}$ ($\vec{A} = -\frac{\mu_0}{4\pi}\int \frac{\vec{J}}{\mathscr{R}}\, d\tau$)

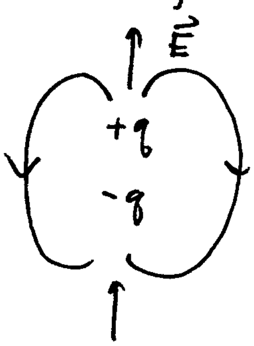
The B.C.'s on $\vec{A}$ + $\vec{B}$ will come in next chapter, when we solve for $\vec{B}$ in materials with currents on boundaries (very analogous to Ch. 3 where we found $\vec{E}$ in materials with free charges on boundaries)

LAST TOPIC of ch. 5: Magnetic Dipoles + Multipoles.

First, consider $\vec{B}$ from small current loop



It's a simple + familiar pattern.
It looks a lot like $\vec{E}$ from an
electric dipole
(at least, outside!)



Calculating $\vec{B}$ (e.g. from Biot-Savart) is a pain here...
But surely this is an important $\vec{B}$ field, looks a lot like what you'd expect from an electron in orbit! So we need it!
Here is a case where $\vec{A}$ will be helpful, we'll find $\vec{A}_{\text{DIPOLE}}$
(and then $\vec{B} = \vec{\nabla} \times \vec{A}$ is straightforward)

(For a finite loop, even $\vec{A}$ is rough, so we'll consider limit $a \to 0$)

- This is like finding V (far away) you write a multipole expansion, + the leading term (or two) then dominate (far away)

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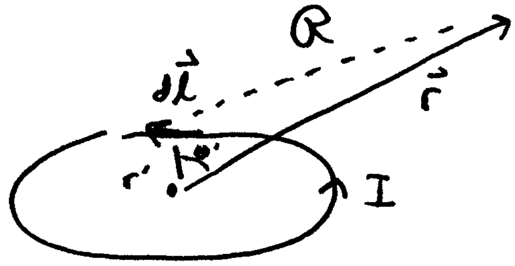


To compute $\vec{A}(\vec{r})$, we just use

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}\,'}{\mathcal{R}} \qquad \text{(Griff 5.64.)}$$

But $\mathcal{R} = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'}$ ← careful, θ' given by $\hat{r}\cdot\hat{r}' = \cos\theta'$

NOT usual polar angle!

Far away, $r \gg r'$, so expand

$$\mathcal{R} \simeq r\left(1 - \frac{r'}{r}\cos\theta' + \mathcal{O}\left(\frac{r'^2}{r^2}\right)\right) \Rightarrow \frac{1}{\mathcal{R}} = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \mathcal{O}\left(\frac{r'^2}{r^2}\right)\right)$$

(Griffiths points out this is $P_1(\cos\theta')$, not a coincidence)

$$\text{so } \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi r}\Bigg[\underbrace{\oint d\vec{l}\,'}_{\text{MONOPOLE}} + \underbrace{\oint \frac{r'\cos\theta'}{r}\,d\vec{l}\,'}_{\text{DIPOLE}} + \cdots\Bigg]$$

~~But the~~. Very reminiscent of multipole expansion for V

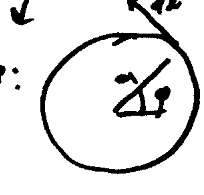
series of terms, each down by one more $\frac{1}{r}$

each has $P_n(\cos\theta')$ dependence.

Here, leading (monopole) term vanishes! $\oint d\vec{l} = 0$.

No "charge" associated with loop.
(MAGNETIC)

do you see why this is $R\,d\vec{l}$ NOT $2\pi R$!?

Look from above:  $d\vec{l} = R\,d\varphi\,(-\sin\varphi, \cos\varphi, 0)$

So for wire loops, the $\vec{A}$ field far away will almost always be dominated by next term ("magnetic dipole term")

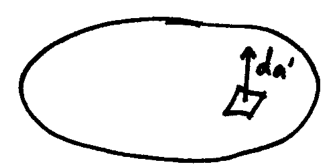
$$\vec{A}(\vec{r}) \simeq \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\vec{\ell}'$$

In principle, can compute this. Note that $\vec{r}$ is "hidden" in $\cos\theta'$, so it can be a royal pain to do this integral. So.. we play a trick:

$$r'\cos\theta' = \vec{r}' \cdot \hat{r}$$

and

$$\oint \vec{r}' \cdot \hat{r} \, d\vec{\ell}' = -\hat{r} \times \iint d\vec{a}'$$



this is not obvious!! Need to use

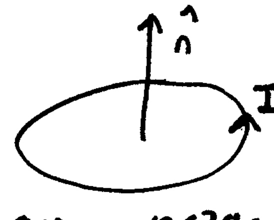
Stokes Theorem ⇒ Prob 1.60e on p. 56 Griff
⇒ Identity 1.108 on p. 57 Griff (using Flyleaf Prod rule 7 and triple prod (1))

This much vector calc manipulation leaves me a little dizzy, but this end result is just too cool! Let's put it together

Define $\vec{m}$ = magnetic moment of current loop $\equiv I \iint d\vec{a}'$

For flat loop, $\iint d\vec{a}' = a\hat{n}$

simple, usual area vector



Direction of $\hat{n}$ determined by RHR and sense of the line integral we started with!

5-45

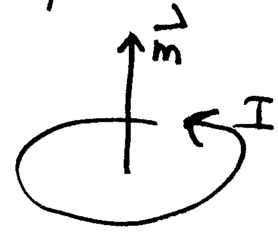
so $\vec{A}(\vec{r}) \simeq \frac{\mu_0}{4\pi r^2} I \iint d\vec{a}' \times \hat{r} = \boxed{\frac{\mu_0}{4\pi r^2} \vec{m} \times \hat{r}}$

Check it out: $\hat{m}$ is a simple property of current source

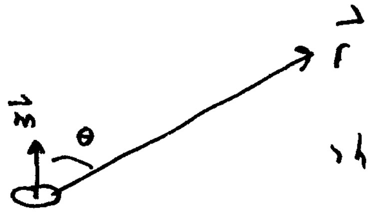
magnetic dipole moment is a vector

points in "area" direction (RHR with $\vec{I}$)

magnitudes is I * area.



If you are at polar angle θ, with $\vec{m} = m\hat{z}$

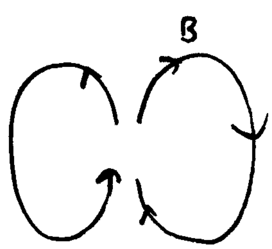


then $\vec{A} = \frac{\mu_0 m}{4\pi r^2} (\sin\theta) \underbrace{\hat{z} \times \hat{r}}_{\hat{\varphi}}$!

so $\vec{A}$ "points" in $\hat{\varphi}$ direction, just like $\vec{I}$ itself.

Potential $\sim \frac{1}{r^2}$, as expect for dipole

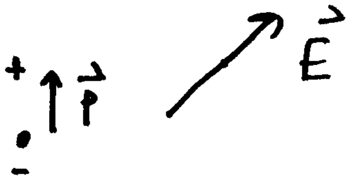
$\vec{B} = \vec{\nabla} \times \vec{A}$ use front flyleaf in spherical, it looks just like $\vec{E}$ dipole!



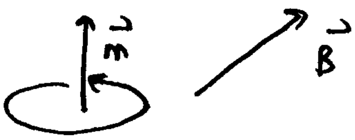
So if you have current loops (small +/or far away) we know $\vec{A}$ and $\vec{B}$ from them. If have <u>bunch</u>, can talk about $\vec{m}$ / unit volume = Magnetization $\vec{M}$.

(A <u>lot</u> like $\vec{P} = \dfrac{\text{electric dipole moment}}{\text{unit volume}}$ = Polarization in ch. 4)

Analogues are very strong!

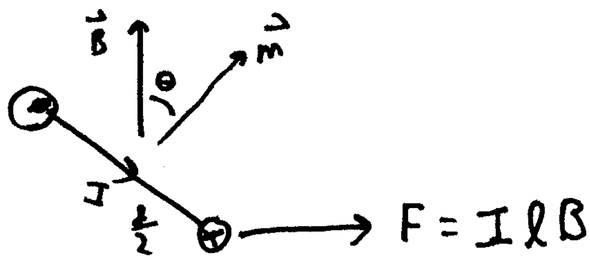
E.g. 

$\vec{\tau}$ on E-Dipole $= \vec{p} \times \vec{E}$



$\vec{\tau}$ on mag-Dipole $= \vec{m} \times \vec{B}$

Easy to show, just use $\vec{F} = I\, d\vec{l} \times \vec{B}$. Easier for <u>square</u> loop, but result is general.



$\therefore \tau = \frac{l}{2} F \cdot \sin\theta$

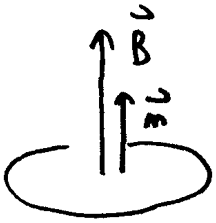
<u>Doubled</u> for two wires feeling torque

$\therefore \tau = 2 \cdot \frac{l}{2} \cdot IlB\sin\theta$

$= Il^2 B \sin\theta$

$= mB\sin\theta$ ✓.

So, mag dipoles tend to "line up" with $\vec{B}$:



Also, remember $\vec{F}_{el} = \vec{\nabla}(\vec{p}\cdot\vec{E})$

now we have $\vec{F}_{mag} = \vec{\nabla}(\vec{m}\cdot\vec{B})$.

so, mag dipoles get "sucked in" to region of stronger $\vec{B}$ field (if they were already "lined up" by torque, that is)

[• Mag Dipoles + Elec dipoles have similar fields, "potential", torque, + force,.. What you know about $\vec{p}$, $\vec{P}$, etc will help... but there are also many differences (Dots tend to become x's in many places, $\vec{B}$ is always subtler, $\vec{A}$ is very much more complicated...]

What's next (Ch. 6) is to parallel Ch. 4 ...

Apply $\vec{B}$; matter adjusts, it polarizes, or in this case, it $\boxed{\text{Magnetizes}}$. You get an $\vec{M}$ (which itself creates $\vec{B}_{induced}$)

(And, new feature, direction of $\vec{M}$ is not always same as $\vec{B}$!)

we need to figure out what $\vec{B}_{induced}$ from $\vec{M}$ looks like, and what $\vec{M}$ is created by $\vec{B}_{ext}$, + we're golden!