✯ ZEN & THE ART OF MULTIPOLE EXPANSION ✯

Separation of variables and multipole expansion are two sides of the same cosmic coin

Potential from a Segment of Charge

A short uniform line charge density λ extends from the origin to the point (0,0,-d).

Finiding V(**r**) everywhere (exactly) is a nasty problem (because curly R is a problem)!

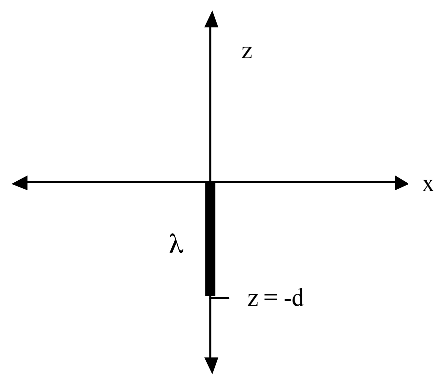
But, finding V(z) just on the z-axis is straightforward. (You worked it out on the exam!)

$$V(z) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\vec{r}\,')d\tau'}{\mathfrak{R}}$$

$$\underset{\text{if } z>0}{=} \frac{1}{4\pi\varepsilon_0}\int_{z'=-d}^{z'=0} \frac{\lambda dz'}{(z-z')}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}(-)\ln(z-z')\big|_{-d}^{0}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{z+d}{z}\right)$$



Look over the lines above (now or at home), convince yourself that you *could* have worked that out! (The condition z>0 in line 2 just lets me safely argue that |z-z'| = z-z')

We'll USE this result to get a quick and simple approximation for V(r) far away from the origin! When you're "far away", d/z <<1. So, let's define $\varepsilon \equiv \frac{d}{z}$. We have:

$$V(z) = \frac{\lambda}{4\pi\varepsilon_o}\ln(1+\varepsilon) \approx ??$$

In general, Taylor's theorem says (for any function f(x)):

$$f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0) + \frac{\varepsilon^2}{2!}f''(x_0) + \ldots$$

With f(x)=ln(x), and x_0=1, I claim (please go home and CHECK THIS for yourself!)

$$\ln(1+\varepsilon) \approx 0 + \varepsilon - \frac{\varepsilon^2}{2} + \ldots$$

Use this to write the first two non-zero terms of V(z) (in terms of z, rather than ε)

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2) Consider the charge distribution on the other side of the page. This problem does **not** have spherical symmetry. Why can I still argue that the potential has the familiar form:

$$V(r,\theta)=\sum_l \left(A_l\cdot r^l+\frac{B_l}{r^{l+1}}\right)\cdot P_l(\cos\theta)?$$

(Specifically, *in what regions of space* does the solution look like this?)

3) The final trick here is that this "general" potential must MATCH your answer (on the bottom of the other side of the page) when $\theta = 0$ (i.e. points on the z-axis)!

USE that to figure out the first couple of A's and B's.

(Do any terms vanish?)

$$P_0(x)=1$$
$$P_1(x)=x$$
$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2}$$
$$P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x$$

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Part 2 – Multipole expansion

A potential from localized charges can be expanded into the form:

$$V = \frac{1}{4\pi\varepsilon_o}\left(\frac{"monopole"}{r} + \frac{"dipole"}{r^2} + \frac{"quadrapole"}{r^3} + ...\right)$$

i. For the problem you've been working on, what are the monopole and dipole moments? Do these answers make sense physically?

ii. Which terms would change if the charge distribution were shifted up by d/2, so that it was centered on the origin?

iii. Does your answer to part ii make sense physically? What is the physical significance of the dipole term when there is a net charge?

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