Homework Set #11, Physics 3320, Fall 2014, Due Wednesday, Nov 19 (start of class)

1. Let's build some familiarity with the vector potential **A** by using it to compute something we already know: the B-field of a long straight wire carrying a steady current I. (See Griffiths Ch 5.4.1 to review the formalism!) But, if you try to compute the vector potential at a distance s from an *infinitely-long* straight wire, you will get a nasty surprise: the integral for **A** diverges logarithmically. There is a nice trick for getting around the infinity in such cases:
A) Compute the vector potential $\mathbf{A}(s)$ at a distance s from the center of a *very long* wire, so the limits of integration are $+L$ to $-L$ (where $L >> s$), instead of $-\infty$ to $+\infty$.
B) Compute **B** by taking the curl of **A** you got above, and only *then* (at the end!) taking the limit as $L \to \infty$. (When computing the curl, choose coordinates intelligently. Should you work in Cartesian coordinates, spherical coordinates,…?)

2. At the start of Ch 10, Griffiths walks us through the reasoning behind Eq 10.2 and 10.3, which are the key formulas relating **E** and **B** to V and **A.**
A.) Substitute these relations into Maxwell's equations and show that the general equations for the potentials $V(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ (without using any particular gauge condition) are

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2 \mathbf{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$

Note that I DO this in my lecture notes for you, as does Griffiths. The idea here is for you to do it yourself, explain your work, convince yourself and the grader! I strongly suggest you close the book (though maybe not the front flyleaf) and derive it on your own as much as possible!
The equations in a particular gauge can be obtained from these general equations by substituting in the gauge condition:
- Derive the differential equations for V and **A** in i) the Lorentz gauge and ii) in the Coulomb gauge by this method.
(Note: In the Lorentz gauge $\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = 0$*, whereas in the Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$*. Again, this is done in the lecture notes and the text, we just want you to go through it for yourself!)*

B. Consider a point charge at rest at the origin. You know $V(\mathbf{r}, t) = q / (4 \pi\varepsilon_0 r)$, and $\mathbf{A}(\mathbf{r}, t)=0$. (Right? Convince yourself, show us that these yield the familiar fields for such a charge.)
- Explicitly check to see if we are in the Coulomb gauge, the Lorentz gauge, or perhaps, both, or maybe neither!
- Now, introduce a gauge transformation $A_{new}=A+\nabla f$, $V_{new}=V-\partial f/\partial t$, using the particular choice $f(\mathbf{r},t) = q\, t/(4 \pi\varepsilon_0 r)$. Find the new V and **A**. Briefly discuss your results: are the potentials "static" or time dependent? Does this represent the same physical situation, or is something different here? Are we in the Coulomb gauge, or the Lorentz gauge now? (Do we *have* to be in one of those?) What has changed, what is the same?

C. Look at Griffiths Example 10.1. Explicitly check to see if he is in the Coulomb gauge, the Lorentz gauge, or perhaps, both, or maybe neither

3. Say the potentials throughout space and time are V(**r**, t) = 0 and A(**r**, t) = A_0 sin(k(z-ct)) $\hat{x}$, where A_0 and k are given constants, and c is the speed of light.
- Find the **E** and **B** fields everywhere in space and time, and comment on the *physics* here, what have we got going on?
- Are we in the Coulomb gauge, the Lorentz gauge, both, or neither?
- Use the formulae from Question #2 to find ρ and **J** here. Does the answer agree with your intuition about the physics of this question?
- Is it possible, in principle, to find a different gauge for this problem in which **A**(**r**,t) =0? (If so, find the gauge transformation. If not, why not?)

4. Consider a problem similar in spirit to Griffiths Example 10.2 (which will be very helpful to study before working this one!) but instead of an infinite straight wire, we have an infinite SHEET which lies is the xy plane. The surface current K(t) = 0 for t<=0, but at t=0, suddenly the surface current turns on, so it is a constant K_0 in the +x direction, instantly, and everywhere in this plane. (There is no *charge* density anywhere).

A) Find the scalar and vector potentials everywhere in space and time, and thus the E and B fields at a distance z above (or below) the xy plane at time t. Sketch or draw these fields, and describe them in words. Briefly, comment on them. (Does your answer make sense, is it e.g. what you would have expected from phys 3310? I find *part* of the answer VERY surprising!)
Note: You may find it helpful to remember that $\sqrt{z^2} = |z|$

B) I claim that your answer to part A is really an "outgoing plane wave". Show/convince us that this is the case. What direction is the wave propagating? At what speed?
(Note: this is the first case where we have derived the origin of a traveling wave resulting from the "turn-on" of a current. It's a primitive model of an antenna! More on this in Ch 11 soon)

C) Show explicitly that B is *not* continuous at the (z=0) xy plane, but that it satisfies the appropriate *boundary condition* there.
Show explicitly that E *also* satisfies the appropriate boundary condition at the (z=0) xy plane.

Extra Credit:
Back early in Chapter 5.4.1, Griffiths shows that (in statics), you can ALWAYS pick a vector potential whose divergence is 0. (The Coulomb gauge) **Show that** in general, it is always possible to choose $\nabla \cdot \mathbf{A} = -\mu_0 \varepsilon_0 (\partial V / \partial t)$ (i.e. the Lorentz gauge).
You may assume that you know how to generically solve PDEs of the form Griffiths Eq 10.16 (these are the last equations in section 10.1.3), because in fact there is a general integral expression to solve those PDE's! Note that my discussion of this question in my Chapter 10 lecture notes was incorrect!)

And, a second question – in general, can you always find a gauge where V(**r**,t)=0? If so, tell us how to do it. If not, why can't you?