3320 10.1

Maxwell's Eq'ns and potentials

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = \underline{-\partial \vec{B}/\partial t}$$
$$\nabla \times \vec{B} = \mu_0 \vec{J} + \underline{\frac{1}{c^2}\partial \vec{E}/\partial t}$$

- The underlined $\left(\frac{\partial}{\partial t}\right)$ terms are new. This term: dynamics!
- We found some sol'ns (traveling waves) in empty space. But, this is not fully general.

Given $\rho(\vec{r},t)$, & $\vec{J}(\vec{r},t)$, you'd like to find $\vec{E}$ & $\vec{B}$ (6 unknowns!)

What did we do in E&M I, without those $\partial/\partial t$ terms?

1) "Direct solution", Coulomb's Law gives $\vec{E}$ from ρ
Biot Savart gives $\vec{B}$ from $\vec{J}$

2) "Use Potentials". There are fewer unknowns (V & $\vec{A}$),

In electrostatics, $\vec{E} = -\vec{\nabla} V$, and $\nabla^2 V = -\rho/\varepsilon_0$

Solution was $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(r')\, d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$ Look familiar?

In magnetostatics, $\vec{B} = +\vec{\nabla}\times\vec{A}$, and $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

Solution was $\vec{A}(r) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(r')\, d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$

(You probably spent much more time on V than $\vec{A}$, so it may be more familiar, but all this is Ch. 2, 3, + 5)

Alas, these simple results do not quite work if ρ or $\vec{J}$ depend on time. But we can patch things up: potentials are still useful!

Math fact ①: If $\vec{\nabla}\times\vec{E}=0$, then there exists a scalar $V(\vec{r})$ such that $\vec{E}=-\vec{\nabla}V$ (Can you convince yourself why?)

(Alas, $\vec{\nabla}\times\vec{E}\neq 0$ anymore, if $\partial B/\partial t\neq 0$!) [But this math fact holds for any vector function whose curl is zero]

Math fact ②: If $\vec{\nabla}\cdot\vec{B}=0$, then there exists a vector fn $\vec{A}(\vec{r})$ such that $\vec{B}=\vec{\nabla}\times\vec{A}$.

So in dynamics, $\vec{\nabla}\cdot\vec{B}=0$ still, and thus $\vec{A}$ still exists & is helpful.

Faraday says $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\vec{\nabla}\times\left(\frac{\partial\vec{A}}{\partial t}\right)$

so $\vec{\nabla}\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$. Now look back at "fact ①" above:

If this is true, then there exists a scalar $V(\vec{r})$ such that

$$\vec{E}+\partial\vec{A}/\partial t=-\vec{\nabla}V$$

Summary: In electrodynamics, there exist $V(r,t)$ and $\vec{A}(\vec{r},t)$ with

$$\boxed{\begin{aligned}\vec{B}(\vec{r},t)&=\vec{\nabla}\times\vec{A}(\vec{r},t)\\ \vec{E}(\vec{r},t)&=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}}{\partial t}(\vec{r},t)\end{aligned}}$$

[This is new. Note, it's the same $\vec{A}$ in both eq'ns.

3220 10.3

The vector potential plays a more central role now, it's needed for both $\vec{B}$ and $\vec{E}$! OK, so how do you *find* V and $\vec{A}$?

One way is to use Maxwell's eq'ns, eliminating $\vec{E}$ + $\vec{B}$ for V and $\vec{A}$.

E.g. $\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \Rightarrow \vec{\nabla}\cdot\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right) = \rho/\varepsilon_0$

so $-\nabla^2 V - \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{A} = \rho/\varepsilon_0$ (*) Eq'n #1

E.g. $\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}} \Rightarrow \nabla\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$

So $\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} - \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$ (*) Eq'ns #2.

This is a vector eq'n, so really it's 3 eq'ns.

Those (4) *'d equations are PDE's, you could in principle use them to find V & $\vec{A}$ given ρ + $\vec{J}$.

- We've got 4 unknowns (not 6). That's progress.

Still, these eq'ns do not look pretty. (They aren't!) But, we can simplify...

(The key is to take advantage of something we talked about in E+M I, the fact that more than one "$\vec{A}$" can work for us: gauge freedom)

The key idea: If $\nabla\times\vec{A} = \vec{B}$ is "given", and $\nabla V = -\vec{E}$ is "given", $\vec{A}$ & V are not unique! There are many (∞'ly!) different V's & $\vec{A}$'s that give the same $\vec{B}$ & $\vec{E}$! This freedom in V & $\vec{A}$ is called

gauge freedom

Ex: Adding a constant to V or a constant vector to $\vec{A}$ changes nothing physical. But, gauge freedom is much richer than just "constant shifts"!

Math fact: $\vec{\nabla}\times(\nabla f) = 0$ for any function $f(\vec{r},t)$ you want.
(Convince yourself, it's a quick proof). So if you have some $\vec{A}_{old}$ & V_{old}, + then invent $\vec{A}_{new} = \vec{A}_{old} + \nabla f$ ← Any scalar fn in the universe!

then $\vec{B}_{new} = \vec{\nabla}\times\vec{A}_{new} = \vec{\nabla}\times(\vec{A}_{old} + \nabla f) = \nabla\times\vec{A}_{old} + \vec{\nabla}\times(\nabla f)$

$= \vec{B}_{old} + 0$, always!

So this $\vec{A}_{new}$ is "equivalent" as far as the physical $\vec{B}$ is concerned.

This is nice - it means you can alter $\vec{A}$ ("gauge transform $\vec{A}$") a lot, by adding $\vec{\nabla}$(anything), + still leave the physics ($\vec{B}$) unchanged.

In particular, the game will be to pick our f cleverly, so that $\vec{A}_{new}$ has some nicer, desired property that $\vec{A}_{old}$ didn't have.

3320 10.5

<u>Possible</u> <u>problem</u> : Recall that $\left\{\begin{array}{l}\vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{E} = -\nabla V - \partial\vec{A}/\partial t\end{array}\right.$

If we altered $\vec{A}_{old}$ by adding $\vec{\nabla}f$ to it, keeping $\vec{B}$ unaffected, we still might change $\vec{E}$. That's no good! Can we avoid this? Yes! The trick is that <u>if</u> you alter $\vec{A}$, you <u>must</u> also alter V : How?

$$\vec{E}_{old} = -\nabla V_{old} - \partial\vec{A}_{old}/\partial t$$

$$\vec{E}_{new} = -\nabla V_{new} - \partial\vec{A}_{new}/\partial t = -\nabla V_{new} - \frac{\partial}{\partial t}(\vec{A}_{old} + \vec{\nabla}f)$$

But I want (need!) $E_{old} = E_{new}$, changing "gauge" must not change <u>physics</u>!

so $\vec{E}_{old} = \vec{E}_{new} \Rightarrow -\vec{\nabla}(V_{new} - V_{old}) - \frac{\partial}{\partial t}(\vec{A}_{old} + \nabla f - \vec{A}_{old}) = 0$

thus $-\vec{\nabla}(V_{new} - V_{old} + \partial f/\partial t) = 0$ Ahh... that's the trick.

If I say $\boxed{\vec{A}_{new} = \vec{A}_{old} + \nabla f}$

<u>and</u> $\boxed{V_{new} = V_{old} - \partial f/\partial t}$

<u>Together</u> this keeps $\vec{E}$ & $\vec{B}$ the same. You need <u>both</u>, with the same f! This is called a "gauge transformation".

Again, f can be <u>any</u> function $f(\vec{r}, t)$ you want. There is a <u>lot</u> of freedom to <u>alter</u> $\vec{A}$ (but, you're constrained to alter V as shown above). In doing so, remember that $\vec{E}$ & $\vec{B}$ are unaltered.

- The <u>physics</u> is not changed when you <u>change</u> <u>gauges</u> (pick a new f)

3320 10.6

Look back at our big eq'ns for V & $\vec{A}$ on p. 3 :

$$-\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\varepsilon_0$$

$$-\nabla^2 \vec{A} + \nabla(\vec{\nabla}\cdot\vec{A}) = \mu_0 \vec{J} - \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right) - \frac{1}{c^2}\partial^2\vec{A}/\partial t^2$$

Suppose some $\vec{A}_{old}$ & V_{old} satisfy these. What if (* see next page, you always can!!) I could find a clever f that made $\vec{\nabla}\cdot\vec{A}_{new} = 0$.

This is called "picking a gauge". If I could do this, then the pair of eq'ns above would look much simpler for $\vec{A}_{new}$ & V_{new} :

$$-\nabla^2 V_{new} = \rho/\varepsilon_0$$

$$-\nabla^2 A_{new} + \frac{1}{c^2}\frac{\partial^2 \vec{A}_{new}}{\partial t^2} = \mu_0 \vec{J} - \frac{1}{c^2}\nabla\frac{\partial V_{new}}{\partial t}$$

The upper eq'n is familiar, it's Poisson's eq'n. Well, sweet! We spent a semester learning how to solve that PDE! In fact, the sol'n is

Coulomb's law for voltage: $V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\iiint \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}\,d^3\vec{r}'$ // Done!

So we call this "gauge choice", where $\vec{\nabla}\cdot\vec{A} = 0$, "Coulomb's gauge"

Coulomb's gauge is common in $\left\{\begin{array}{l}\text{Magneto +}\\ \text{Electrostatics}\end{array}\right.$. (But the 2nd eq'n for $\vec{A}$ is still nasty, so it's not used so commonly when you have time dependence.)

3320 - 10.6 b

In the Coulomb gauge, $\nabla^2 V = -\rho/\varepsilon_0 \Rightarrow V = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}\,',t)}{|\vec{r}\,'-\vec{r}|} d^3\vec{r}\,'$

Unfortunately, V alone is not enough any more!
$\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t$, so you also need $\vec{A}$. (And alas, our $\vec{A}$ eq'n is "coupled" & nasty ... a slight bummer)

Also, $V(\vec{r},t)$ depends on $\rho(\vec{r}\,',t)$, at the same instant in time, but all over the universe. That's just odd: it seems to violate our (relativistic) intuition that something here (voltage $(\vec{r})$) should not depend on charges in Alpha Centauri now. (If I care about charges on α-Centauri, surely it's where they were "speed of light travel time" ago!)

Turns out this intuition is correct regarding physical observables here, but $V(\vec{r},t)$ is not a physical observable. $\vec{E}$ is, but it depends on $\vec{A}$ (which it turns out will depend on ρ in just such a way that only past ρ's on α-Centauri matter!) This is all much more sensible in another gauge, so we'll soon abandon Coulomb's gauge!

* What about the "What if" on the previous page? If you have $\vec{A}_{old}$, can you really be sure there exists an f that will ensure $\vec{\nabla}\cdot\vec{A}_{new}=0$?

Well, $\vec{\nabla}\cdot\vec{A}_{new} = \vec{\nabla}\cdot(\vec{A}_{old}+\nabla f) = \vec{\nabla}\cdot\vec{A}_{old} + \nabla^2 f$

So if you want $\vec{\nabla}\cdot\vec{A}_{new}=0$, this amount to asking "can I always find a function f that satisfies $\nabla^2 f = -\vec{\nabla}\cdot\vec{A}_{old}$." ?

Well, sure! That's still just Poisson's eq'n: $\nabla^2 f =$ Something given. That's always solvable, we gave a formula on the previous page. Heck, nature solves it whenever you set up some charges in a given pattern. So yes, such an f will exist, we can always demand $\vec{\nabla}\cdot\vec{A}_{new}=0$

Will I have to find this f? (What a pain, solving a new Poisson problem??)
No, in general if you already knew $\vec{A}_{old}$, you'd be happy, you've already solved the problem! The point here is that we know in principle, up front, that we could shift to a gauge where $\vec{\nabla}\cdot\vec{A}_{new}=0$. If you just start from there, & find $\vec{A}_{new}$ directly (from the simplified PDE's) then we find $\vec{A}_{new}$ from simpler eq'ns, + we're done!

"Picking a gauge" doesn't mean "finding that f". It means "Setting $\vec{\nabla}\cdot\vec{A}$ to be something convenient" to help us find $\vec{A}_{new}$ + V_{new} directly. (E.g in Coulomb gauge, $\nabla\cdot\vec{A}=0$ makes finding V simple)

3320 10.8

Here's a better gauge choice for many (not all!) electrodynamics problems. It's called "Lorentz gauge": I claim you can always pick an f so that

$$\boxed{\vec{\nabla} \cdot \vec{A} = -\frac{1}{c^2}\,\partial V/\partial t}$$

If $\vec{A}_{old}$ doesn't already satisfy this, you'd need an f such that

$$\vec{\nabla} \cdot (\vec{A} + \nabla f) = -\frac{1}{c^2}\frac{\partial V_{new}}{\partial t}, \quad \text{i.e.} \quad \nabla^2 f = -\frac{1}{c^2}\frac{\partial V_{new}}{\partial t} - \vec{\nabla} \cdot \vec{A}_{old}$$

$= \underline{\underline{\text{some}}}$ function

Once again, Poisson always has a sol'n, f exists.

We don't need it, we won't solve for it. We just know it's always possible to pick it to ensure $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\,\partial V/\partial t = 0$ // Lorentz gauge condition

So those main PDE's back on p. 6 now simplify to

$$-\nabla^2 V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2 \vec{A} + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{J}$$

check for yourself! (Just plug the Lorentz condition into the PDE's at top of p. 6)

These aren't so bad. They are decoupled from each other.

They are "wave equations with sources", + there are known sol'ns.

We'll write the general sol'n down on the next page...

Note: $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is called the D'Alembertian operator, + denoted by $\Box^2$ sometimes.

3320 10.9

There is an elegant general sol'n to the wave eq'n with sources.

Griffiths proves it in one page, p. 424

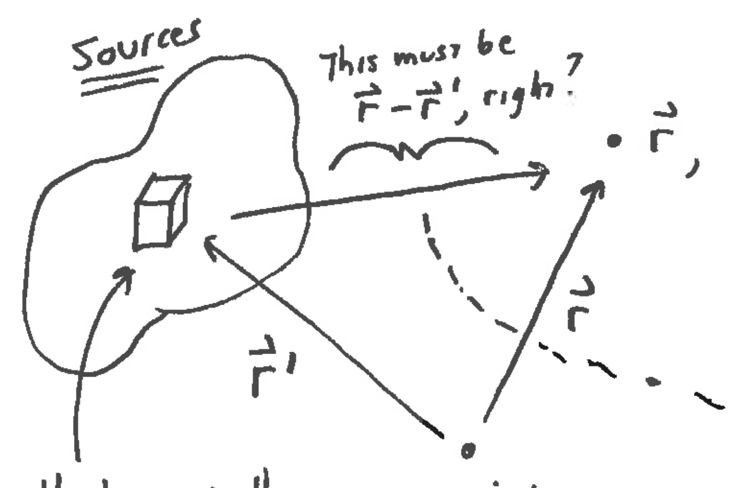
$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0} \iiint \frac{\rho(\vec{r}\,', t_R)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}(\vec{r}\,', t_R)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

They look familiar! Almost our usual old 3310 sol'n to Poisson's eq'n.

The twist here is that "t_R" inside the sources on the right side.
That is not the t on the left side! $t_R \equiv t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ = "retarded time".

Just as we suspected, V "here + now" depends on charges elsewhere as they were earlier. How much earlier? Speed-of-light travel time!



$\vec{r}$, this is where I want to know V & $\vec{A}$ (+ thus $\vec{E}$ & $\vec{B}$)

Here's a small chunk of source for ρ or $\vec{J}$, located at $\vec{r}\,'$.
We'll sum (integrate) over $d^3\vec{r}\,'$ to pick up all sources

It takes light a time = (this distance)/c to get to me.
That's a time $|\vec{r}-\vec{r}\,'|/c$, precisely the retardation we subtract from time t when integrating over the sources.

3320 10.[redacted]10

Notation is a bit fierce, but this makes good sense. $\vec{E}$ & $\vec{B}$ here, now depend on ρ & $\vec{J}$ elsewhere back in time, by speed-of-light travel time.

In principle, we're good here. We have eq'ns for $V(\vec{r},t)$ & $\vec{A}(\vec{r},t)$ + thus $\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ & $\vec{B} = \vec{\nabla}\times\vec{A}$, we're done

Those integrals may be tough to compute analytically, but in principle we just grind 'em out... I'll do one example (see after next pages).

What's next? Griffiths proceeds to manipulate these. I won't pursue the details, but a few comments:

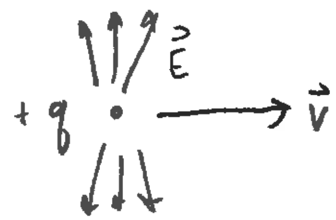
1) You can bypass V & $\vec{A}$, + go straight from ρ + $\vec{J}$ to $\vec{E}$ & $\vec{B}$. It's like the generalization of Coulomb's Law + Biot-Savart. They're called Jefimenko's Eq'ns.

- Elegant, compact, formal. Maybe hard to compute analytically, but not a real issue numerically. So, use 'em if you need 'em in lab!

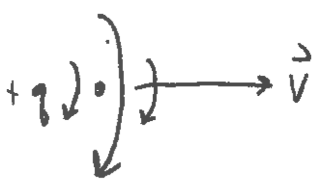
2) If one point charge q moves around in a known way, you know $\rho(\vec{r},t)$ & $\vec{J}(\vec{r},t)$, + can thus find V & $\vec{A}$ as shown above. This is the "Liénard-Wiechert" sol'n. I'd say this is conceptually cool, & if you know sol'ns for one charge, superposition gives you V & $\vec{A}$ for arbitrary (but known) motions of charges!

3320 10.11

The Liénard-Wiechert math is a little ugly, + I'm not sure how often you'd really use it (you need to know the motion a priori, remember!) The result for steady $\vec{v}$ is something we'll get back to using relativity:



$\vec{E}$ field from steady motion of q is vaguely Coulomb-like, but not $\hat{r}/r^2$! And, as $v \to c$, the field lines "concentrate" to be more transverse.



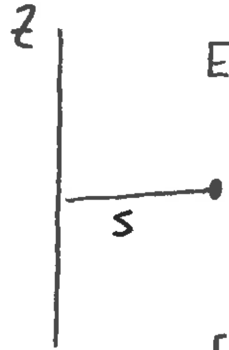
$\vec{B} = \frac{1}{c^2}\vec{v}\times\vec{E}$ here, looks a lot like the relation of $\vec{B}$ to $\vec{E}$ in traveling EM waves!

It's the "right hand rule" configuration you'd guess, but dies off differently than you'd guess, as does $\vec{E}$.

So, we're going to skip on to ch. 11 next, to investigate in more detail the $\vec{E}$ & $\vec{B}$ fields from moving (+ accelerating!) charges, focussing on fields + energy flow. At last, we'll see how one generates those EM waves we've talked about so much!

But first, one example of the "retarded time" potential story!

3320 10.11



Example: A wire sits on the z-axis. At $t=0$, a current I_0 instantly appears everywhere along the wire.

you are here $(\vec{r}, t)$, or $\left[\begin{matrix}(s,0,0,t) \text{ in} \\ \text{cylindrical}\end{matrix}\right]$

What's V, $\vec{A}$ (& thus $\vec{E}$, $\vec{B}$) here?

V is easy, because $\rho = 0$ everywhere at all times

So $V(\vec{r},t) = \iiint \frac{\rho(r', t_R)\, d^3\vec{r}\,'}{4\pi\varepsilon_0 |r - r'|} = 0.$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I(z, t_R)}{|\vec{r} - \vec{r}\,'|}\, dz \;\hat{z}$$

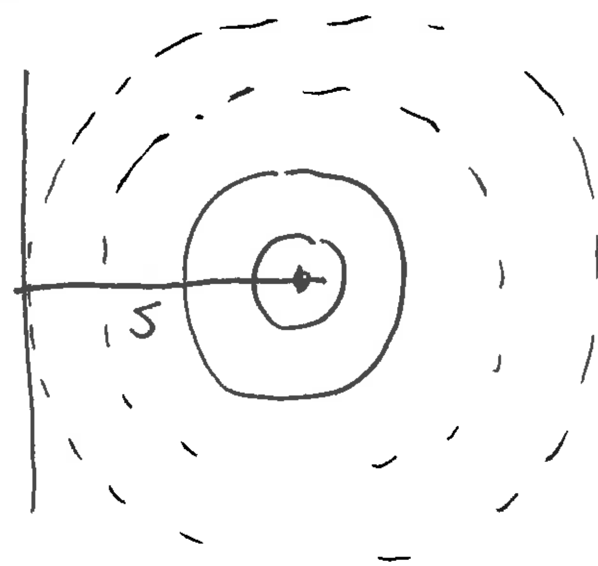
(Note: Since $\vec{J}$ is a 1-D current, our $\iiint \vec{J}\, d^3r' \longrightarrow \int \vec{I}\, dz$)

Now, $I(z, t_R) = I(t_R)$ because I doesn't depend on z. It's just I_0 for all z, or 0. It does depend on time, though. It turns on at $t=0$

At your location, physically, you can only know about currents which existed earlier + "sent" information to you at speed c. So, e.g. even at time $t=0$, no current (which just turned on) could have sent info that reached you yet.

My first clue of anything happening will be at $t = s/c$, when you first notice just the new current here →

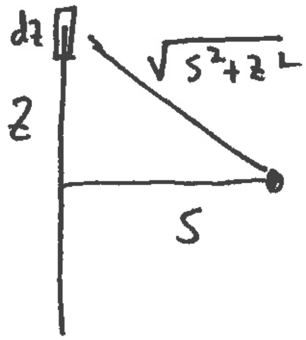
The current elsewhere takes even longer to "get" info to you, based simply on distance.



Circles looking "back in time" for information

3320 10.12

So here's the formal equation:



$$\vec{A}(s,t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I\left(t - \frac{\sqrt{s^2+z^2}}{c}\right)}{\sqrt{s^2+z^2}}\, dz\ \hat{z}$$

notice, z lurks in here!

this is the "retardation" term, (distance/c)

Now, $I(t) = 0$ if $t \leq 0$

So if the argument of $I < 0$, you're integrating $\underline{\underline{0}}$.

So you get 0 for all {times & z's} where $t - \frac{\sqrt{s^2+z^2}}{c} < 0$, i.e. $c^2t^2 < s^2+z^2$

So you get $\underline{\underline{0}}$ if $z^2 > c^2t^2 - s^2$. This means you can cut off the z integration at $z = \sqrt{c^2t^2-s^2}$, beyond that there's no contribution!

(That's the physical argument from the previous page)

$$\text{So } \vec{A}(s,t) = \frac{\mu_0}{4\pi}\int_{-\sqrt{c^2t^2-s^2}}^{\sqrt{c^2t^2-s^2}} I_0 / \sqrt{s^2+z^2}\, dz\ \hat{z}$$

Griffiths takes it from here! Look @ his sol'ns! (At large times you get back our old static results, $\vec{B} \to \frac{\mu_0 I_0}{2\pi s}\hat{\varphi}$, $\vec{E} \to 0$)