Our last example gave $\vec{E}$ & $\vec{B}$ (or V & $\vec{A}$) when we "turned on" current in a wire. This is one example of a more general story: Time varying charge distributions can generate traveling EM waves. We've studied those waves, but not where they come from!

If you think about energy flow, Poynting's theorem tells us that EM power flowing "out to ∞" is given by $\displaystyle\iint_{\text{large surface}} \frac{\vec{E}\times\vec{B}}{\mu_0}\cdot d\vec{a}$

If $\vec{E}$ is static, it goes like $\frac{1}{r^2}$ (or $\frac{1}{r^3}$ if $Q=0$, etc)
If $\vec{B}$ is static, it too drops off like $1/r^2$
] Assuming sources are localized

So in this case, $\iint \vec{S}\cdot d\vec{a}$ vanishes for large surfaces. No radiation

- Turns out steady motion of charges won't generate radiation (defined casually as a flow of energy out to infinity)

We need acceleration of charges to get radiation.

So we're going to focus on the parts of our full EM fields that transmit non-zero power off to ∞. These are the "radiation fields". (We'll ignore other parts, so this is all an approximation, mostly useful far from our localized sources)

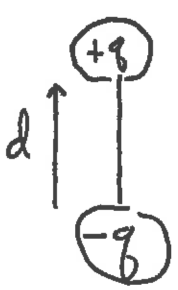
3320 – 11-2

Comment : This class is about classical E&M. Really, we're learning about radiation from macroscopic systems: antennas for radio waves, cavity resonators for microwaves, etc. In general, it is atomic systems that are responsible for most EM radiation (think light bulbs, lasers, etc). For that, you really need QED, a quantum theory. But QED is built from (on) classical E&M. So, despite the limitations, it's worth studying the basics of classical radiation theory. We'll hit just a few select highlights, we won't go deeply into this topic. If you want to learn more about, say, antenna arrays, you'll need to dig further on your own!

To proceed, we'll use our "retarded potential" formalism to find $V(\vec{r},t)$ & $\vec{A}(\vec{r},t)$ from a simple time dependent source.

– We'll work in the Lorentz gauge, it's convenient here.

The one case we'll study in detail is "Electric Dipole Radiation"



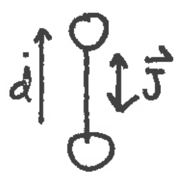
$\vec{P} = q\vec{d}$ depends on time, [e.g simple $\cos(\omega t)$]

[you can imagine how many real macroscopic sources of radiation could be built up by superposition of little dipoles.]

- You might wonder why we don't start from <u>one</u> charge, q, "wiggling around". Turns out (not <u>obvious</u> in advance) that the math is harder, + we'll get most of the key physical insights from our <u>dipole</u>.
- Another case which we won't delve into would be a <u>magnetic dipole</u> 

where a small current <u>loop</u> varies with time. Again, this is physically useful & an obvious "starting case", but here too, there's not much new phys<u>ics</u> to learn (See Griffiths for the details if you like - it really parallels the electric dipole nicely)

For our oscillating dipole, the physics we're going to consider is



a tiny wire which conducts charge back & forth, top to bottom. This is our physical oscillating dipole.

(you might imagine a real antenna as a bunch of these superposed)

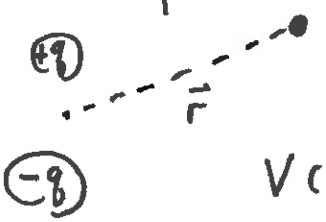
Dipole moment: $\vec{p} = q\vec{d}$; in our case we have simple oscillation,

$$\left.\begin{aligned} \vec{p} &= q_0 d \cos \omega t \ \hat{z} \\ &\equiv p_0 \cos \omega t \ \hat{z} \end{aligned}\right\} \text{ this is our oscillating dipole.}$$

(So $\underline{\underline{\vec{I} = \frac{dq}{dt}\hat{z}}} = \frac{\dot{\vec{p}}}{d} = -q_0 \omega \sin \omega t \ \hat{z}$ is flowing here)

3320 11-4

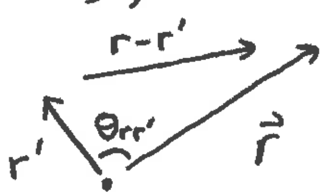
Griffiths follows a path very much like back in Ch 3.4, the Multipole Expansion. The idea back there was that



If you want V (or $\vec{A}$) far away from a dipole,

$$V(\vec{r}) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\text{dipole piece}}{r^2} + \frac{\text{quadrupole stuff}}{r^3} + \cdots\right]$$

[Mathematically, this arose from $\displaystyle \frac{1}{|r-r'|} = \frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \cdots$



$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta_{rr'})$$
]

Remember, we're now after the radiation fields, the parts of the EM fields that don't drop off too fast with r. So we keep only leading multipole terms, + just ignore the rest. We won't know $\vec{E}$ & $\vec{B}$ near the source this way, but that's not what we're after!

I'll take a slightly different approach from Griffiths, let's begin with $\vec{A}(\vec{r},t)$ first, rather than V.

- We'll use our Ch. 10 results in Lorentz gauge $\left(\vec{\nabla}\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}\right)$
- the eq'n for $\vec{A}$ is $\Box^2\vec{A} = \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$
- The sol'n is $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}(r',t_R)}{\mathscr{R}}\,d^3\vec{r}\,'$, with $t_R = t - \frac{|\mathscr{R}|}{c}$

(This is exact & general)

3320 11-5

we are going to make several simplifying approximations!

1) Assume $r >> $ (size of source). This means we're far away.

In our case, with the electric dipole, we're at $r >> d$.

2) Assume $\lambda_{radiation} >> $ (size of source). This is the "small dipole" approximation, $d << \lambda$. In terms of frequency, this means

$$\omega = 2\pi f = \frac{2\pi c}{\lambda} << \frac{2\pi c}{d}, \text{ i.e. } \boxed{d << \frac{2\pi c}{\omega}}$$

Since ω of the EM wave arises from whatever ω characterizes the wiggling of the source, we're focusing on low freq wiggles

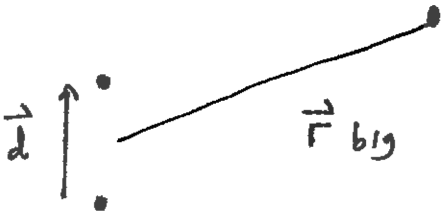
3) We will only keep terms in E & B that fall of no faster than $\frac{1}{r}$ (so it's "radiation"), (so $\iint_{\substack{\text{large}\\\text{distances}}} \vec{S}\cdot d\vec{a}$ is finite)

Formally, this means $r >> \lambda$, (though I haven't shown this)

or together, $\boxed{r >> \lambda >> \text{source size}}$

3320 11-6

Recap of these approximations:



we focus on leading multipole moments.

Since $|\mathcal{R}| = |r - r'|$, we are saying $\mathcal{R} \approx r$

(This is because r' is limited to our <u>small</u> source)

Also $t_R = t - \frac{|\mathcal{R}|}{c} \approx t - \frac{r}{c}$. This says <u>all</u> source points have roughly the <u>same</u> retardation. (We are not neglecting retardation effects, but we're ignoring tiny time differences from different points in the source)

(If ω was too high, this might become a problem, charges would move significant <u>distances</u> in short times, but we're ok from approx #2, $f \ll c/d$)

So $\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r', t - \frac{|\mathcal{R}|}{c})}{\mathcal{R}} \, d^3\vec{r}\,'$

$$\approx \frac{\mu_0}{4\pi r} \int \vec{J}(r', t - \tfrac{r}{c}) \, d^3 r'$$

Recalling that $\vec{J}$ for our dipole example is a 1-D current,

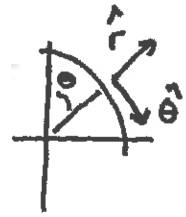
$$\vec{J}(r', t - \tfrac{r}{c}) \xrightarrow[\text{dipole}]{\text{Electric}} \vec{I}(r', t - \tfrac{r}{c}) = -q_0 \omega \sin \omega \underbrace{(t - \tfrac{r}{c})} \hat{z}$$

(see bottom of notes p.3)

3320 11-7

In our approximations, then, $\vec{A}(\vec{r},t) \simeq \frac{\mu_0}{4\pi r} \int_{-d/2}^{+d/2} \underbrace{-q_0 \omega \sin\omega\left(t-\frac{r}{c}\right)}_{\text{None of this depends on } z\,!} dz\, \hat{z}$

$$\left.\begin{aligned} \text{so } \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi r} - q_0 \omega \sin\omega\left(t-\frac{r}{c}\right) d\, \hat{z} \\ &= \frac{\mu_0}{4\pi r} \dot{\vec{p}}\left(t-\frac{r}{c}\right), \text{ if you prefer} \end{aligned}\right\}$$ This is known as a spherical wave potential

Since $\vec{B} = \vec{\nabla}\times\vec{A}$, using spherical coords  $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$ (convince yourself!)

and $\vec{A} = -\frac{\mu_0}{4\pi r} \omega \underbrace{q_0 d}_{(\equiv p_0)} \sin\omega\left(t-\frac{r}{c}\right)\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\varphi} = -\frac{\mu_0}{4\pi} p_0\omega\, \hat{\varphi} \left[\frac{1}{r}\frac{\partial}{\partial r}\Big(r \cdot \overbrace{\frac{-\sin\theta}{r}\sin\omega\left(t-\frac{r}{c}\right)}^{A_\theta}\Big) - \frac{1}{r}\frac{\partial}{\partial\theta}\Big(\underbrace{\frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)}_{A_r}\Big)\right]$$

$$= -\frac{\mu_0}{4\pi}\frac{p_0\omega}{r}\hat{\varphi}\left[-\sin\theta \cdot \left(-\frac{\omega}{c}\right)\cos\omega(t-r/c) \; \cancel{\sin\omega(t-r/c)} + \frac{\sin\theta}{r}\cos\omega(t-r/c)\right]$$

But remember, approx #3 says keep only terms $\sim 1/r$ (not $1/r^2$)

So only the 1st term contributes!

$$\vec{B} \approx -\frac{\mu_0}{4\pi}\frac{p_0\omega^2}{rc}\sin\theta\cos\omega(t-r/c)\,\hat{\varphi}$$

[It's a traveling wave, moving out in radius at speed c!]

That $\omega^2 p_0$ looks like $\ddot{p}$, it's the acceleration of q that gives us our radiation field!

What about $V(\vec{r},t)$, & thus $\vec{E}$? We could go back to

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(r', t_R)}{R}\, d^3\vec{r}\,'$$, this is what Griffiths does.

or we could argue, in Lorentz gauge, $\frac{\partial V}{\partial t} = -c^2\, \vec{\nabla}\cdot\vec{A}$, and use this to deduce V, (that's a cute trick!)

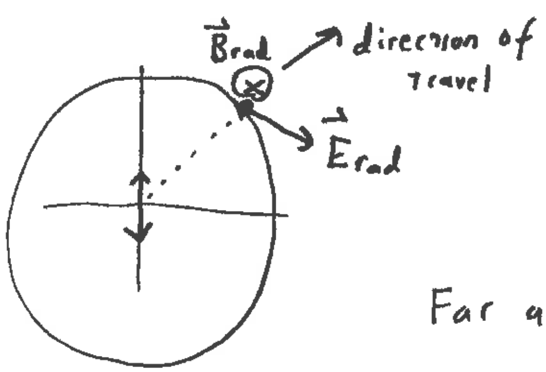
But I'm going to take a slightly cheaty shortcut way out....

We have a simple, time varying, sinusoidal $\vec{B}$ field traveling outwards in a vacuum. This is a simple spherical traveling wave! So, from Ch. 9 I will argue that we know about traveling EM waves:

$\vec{E}$ should be in phase, $\perp$ to both $\vec{B}$ and direction of travel, $\hat{r}$,

i.e. $\vec{E} = c\,\vec{B}_{rad} \times \hat{r}$ (just like always in ch. 9!)

$$= -\frac{\mu_0}{4\pi} p_0 \frac{\omega^2}{r} \sin\theta \cos\omega\left(t - \frac{r}{c}\right)\hat{\theta}$$



This is a simple, monochromatic spherical wave, traveling outwards at speed c.

Far away from source, it'll look like plane waves!

This, then, is where plane waves originate: oscillating dipoles!

3320 11-9

Let's find out about power flow, using Poynting vector:

$$\text{Total Power} = \iint \vec{S}\cdot d\vec{a} = \iint \frac{\vec{E}\times\vec{B}}{\mu_0}\cdot d\vec{a} = \iint \frac{1}{\mu_0}\left(\frac{\mu_0 p_0 \omega^2}{4\pi}\right)^2 \cdot \frac{1}{c r^2}\sin^2\theta \cos^2\omega\left(t-\frac{r}{c}\right) da$$

For a large spherical surface at radius r, $da = r^2 \sin\theta \, d\theta \, d\varphi$

$$\text{Total Power} = \frac{\mu_0 p_0^2 \omega^4}{(4\pi)^2 c}\cdot \cos^2\omega\left(t-\frac{r}{c}\right)\cdot 2\pi \underbrace{\int_0^{\pi} \sin^2\theta \cdot \sin\theta \, d\theta}_{\text{use } \sin^2\theta = 1-\cos^2\theta \text{, this gives } 4/3}$$

convince yourself!

$$\text{Power} = \frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c}\cdot 2\pi \cdot \frac{4}{3}\cdot \cos^2\omega(t - r/c)$$

If you time-average, $\langle \cos^2\omega(t-\frac{r}{c})\rangle = 1/2$, giving

$$\text{Time average power} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \leftarrow$$

Note, again, a more formal derivation shows this arises from $(\ddot{p})^2$, acceleration!

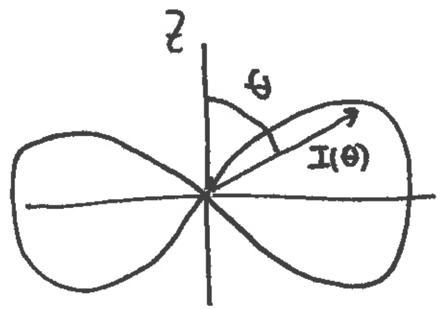
[average Intensity would be this $/4\pi r^2$]

- If we turned on & then off our dipole, this power would ~~[illegible]~~ sweep past us at the retarded time, r/c.
- This radiated power leaves, it must come from somewhere! Someone is pumping this power into that oscillating dipole. This tells me there must be a "radiation reaction force" on the charges in the dipole, arising from these EM fields, that we must work against.

3320 11-10

- Also note that in the step before we integrated over the sphere, our $\langle S \rangle$ had an angular dependence:

$P_{outward} \propto \sin^2\theta$. So, you get no power $\updownarrow$ flowing to the 2 poles!



- Polar plot of Intensity as a function of θ.
- Distance (r) on this plot represents $I(\theta)$
- Power flows out mostly ~ xy plane.

Electrical engineers talk about an effective radiation resistance

This is defined by $R_{rad} \equiv \dfrac{P_{ave}}{I_{rms}^2}$ — It's the equivalent resistance that would dissipate the same energy that's pouring out as radiation.

this is rms current, not intensity

For us, current $= q_0 \omega \sin\omega t$, so $I_{RMS}^2 = q_0^2\omega^2 \cdot \frac{1}{2}$,

and $$R_{Rad} = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \Big/ \frac{1}{2} q_0^2\omega^2 = \frac{\mu_0 \omega^2 d^2}{6\pi c} \underbrace{=}_{\text{using } \omega = cK} \frac{\mu_0 c K^2 d^2}{6\pi}$$

and finally using $c = 1/\sqrt{\mu_0\varepsilon_0}$,

$$R_{rad} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{6\pi}(Kd)^2 = \underbrace{(377\,\Omega)}_{\sqrt{\mu_0/\varepsilon_0} \,=\, \text{"impedence of free space"}} (Kd)^2/6\pi$$

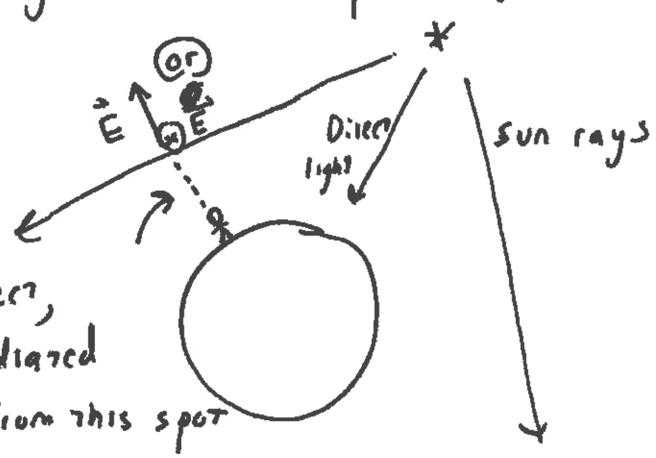
(Recall, in our approximation, $Kd = \frac{2\pi d}{\lambda} \ll 1$, so this R tends to be small.)

(Small dipoles are inefficient radiators)

If we pursued "antenna theory" here, we would see how superposing dipoles to make a real antenna could improve the efficiency. (Real antennas have $d = \lambda/2$, or $\lambda/4$, so our approximations break down, but the full theory gives better, more efficient power, hence "quarter wave" or "half wave" antennas.

The $\omega^4 \propto 1/\lambda^4$ power dependence here is physically important. Higher frequencies radiate more power (just keep $\omega \ll c/r$, though!)
& $\omega \ll c/d$

This is (partly) responsible for the blue sky! Incoming light (with all visible ω's present) excites atoms in sky, making oscillating dipoles. These dipoles (re)radiate (this is called scattering), + you get lots more power from the higher ω (blue!) frequencies.

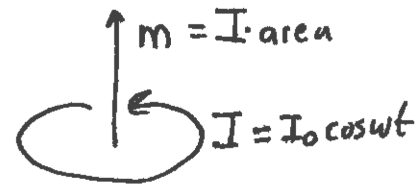


indirect, re-radiated light from this spot reaches you, but I is mostly from blue

Also, due to $\sin^2\theta$ factor, most light reaching you in this indirect way will be polarized ⊥ to the ray shown (into the page for the particular ray I drew)

3320 11-12

Summary / synopsis of remainder of ch. 11.



Griffiths calculates $\vec{E}$ & $\vec{B}$ for a current loop

The story is very similar! You get similar results for $\vec{E}$, $\vec{B}$, & $\vec{S}$

However, you do find that for a given rough "sized" object, the magnetic dipole radiation is weaker than from our electric dipole

$$\frac{I_{mag}}{I_{elec}} \sim \left(\frac{\text{mag moment}}{c \cdot \text{elect dipole}}\right)^2 \sim \left(\frac{I \cdot \text{area}}{c \cdot q_0 d}\right)^2 \sim \left(\frac{I \cdot d^2}{c \cdot (I/\omega) d}\right)^2 \sim \left(\frac{\omega d}{c}\right)^2$$

But recall our approx says $\omega \ll c/d$, so this is <u>very</u> small.

Mag. dipole radiation is smaller than electric dipole radiation

(Electric quadrupole radiation would be smaller still!)

Griffiths next tackles radiation from a single charge, q, moving in some path. Making the now familiar "radiation approximations", that you are <u>far</u> away, + considering components of motion giving $c/\omega \gg$ distance, you again get very similar results.

I indicated the idea earlier: both $\vec{E}$ & $\vec{B} \propto \frac{\ddot{p}}{r}$ (retarded)

(where $\vec{p} = q\vec{r}$). In other words, as claimed, it's the <u>acceleration</u> of charges that gives rise to radiation fields

3320 11-13

There's a general result for total power radiated from moving charges, Larmor's formula. Griffiths derives it, & it agrees with our special case result for electric dipoles. But let's guess it by a method called "dimensional analysis", use units & basic physics. (It's a powerful & often used trick to guess relationships in physics!)

we want radiated power $= W = J/s = kg\, m^2/s^3$.

For a moving charge, what could this power depend on?

we just argued acceleration, a (units m/s^2)

(But not position or velocity, those can't produce radiation)

clearly charge q might matter (units C)

I expect μ_0 and/or ε_0 might enter, since this is EM radiation!

$$[\mu_0] = N/A^2 = kg\, m/C^2$$

$$[\varepsilon_0] = C^2/Nm^2 = \frac{C^2}{kg}\, s^2/m^3$$

what else? I can't think of anything, can you?

So assume Power $= \mu_0^{\,a}\, \varepsilon_0^{\,b}\, q^c\, a^d$, [where a, b, c, d are some unknown #'s]

we have 4 base units & 4 unknowns, we can find them!

Quite amazing, really, the answer here will be unique!!

So $\text{Power} = \frac{\text{Kg m}^2}{\text{s}^3} = \left[\frac{\text{Kg m}}{\text{C}^2}\right]^a \left[\frac{\text{C}^2 \text{s}^2}{\text{Kg m}^3}\right]^b [\text{C}]^c \left[\frac{\text{m}}{\text{s}^2}\right]^d$

Look at Kg: $(\text{Kg})^1 = (\text{Kg})^{(a-b)}$ which says $a-b=1$ *(1)

Look at m: $(\text{m})^2 = (\text{m})^{(a-3b+d)}$ " " $a-3b+d=2$ *(2)

Look at sec: $(\text{s})^{-3} = (\text{s})^{(2b-2d)}$ " " $2b-2d=-3$ *(3)

" " Coul: $(\text{C})^0 = (\text{C})^{(-2a+2b+c)}$ " " $-2a+2b+c=0$ *(4)

Putting (1) into (2) gives $1-2b+d=2$
(3) says $2b-2d=-3$
Adding these says $1-d=-1$ or $d=2$ //

Thus (3) says $2b-4=-3$, or $b=1/2$

then (1) says $a=1+b=3/2$

Finally (4) says $-3+1+c=0$, so $c=2$

Thus $P = \mu_0^a \varepsilon_0^b q^c a^d = \mu_0^{3/2} \varepsilon_0^{1/2} q^2 a^2$

or, using $\frac{1}{c} = \sqrt{\mu_0 \varepsilon_0}$, $P = \frac{\mu_0}{c} q^2 a^2$. Bingo!

Larmor formula says $P = \frac{\mu_0}{6\pi c} q^2 a^2$ //

(we could've used our special case, where $\ddot{p} = q\ddot{x} = qa$ to guess the $\frac{1}{6\pi}$)

This is no derivation, just a cool & broadly usable method to guess functional dependences when you know the basic physical quantities involved.

3320 11-15

Some final comments:

– These E_{rad} & B_{rad} fields are not the whole story. They're just the part of the fields carrying energy away. Eg, the Coulomb field (and other bits for moving charges!) are still present far away. They just don't contribute to the Poynting flux at large r.

– If charges move relativistically, there are corrections! This says to me it's high time we studied special relativity!

– Our dipole was charge neutral. We never worked out a "monopole" radiation. E.g, if a balloon has Q painted on it, & it breathes in & out. ρ certainly has a time dependence, but if you follow Griffiths, you'll find $\vec{E} \propto \hat{r} \times (\hat{r} \times \ddot{\vec{p}})$

This combo would vanish for a breathing mode, where $\ddot{\vec{p}}$ points in $\vec{r}$ direction

there IS no monopole radiation. Radiation arises from Kinks in $\vec{E}$ & $\vec{B}$ lines that propagate outwards. No Kinks (like for breathing balloon) $\Rightarrow$ no radiation. (In a similar way, spherically collapsing stars don't radiate gravitational radiation)