Principles of Relativity

↓

Geometry of space-time (Simultaneity, γ, length contraction, time dilation)

↓

Lorentz transformations

↓

Invariants & "4-vectors"

↓

Relativistic Electrodynamics

3320 12-1

Special theory of Relativity:

"Relativity" refers to a very basic (+ intuitive!) idea about nature. The laws of physics are agreed upon by all inertial observers. This does not mean observers agree on everything! In a classical (Newtonian) world, if you're in a frame moving with respect to me, we will disagree on positions & velocities of objects (for instance), but we'll agree on acceleration, force, & mass, + thus our separate measurements will agree that $\vec{F} = m\vec{a}$.

Even if you just tilt your head, we will disagree on e.g $x, y, \& z$ components of $\vec{F}$ (and $d\vec{r}$), but we'll agree $\text{Work} = \vec{F} \cdot d\vec{r} = \Delta KE$.
(The dot product of vectors is invariant with respect to rotations!)

By the way, I'm talking about inertial reference frames, i.e. "not accelerating". We will agree we're inertial by checking Newton's 1st law (If, in the absence of net external force, objects in our frame persist in their same state of motion, we're in an inertial frame.) It's an experimental fact that anyone moving with constant $\vec{v}$ w.r.t. an inertial frame is also in an inertial frame.

What about Maxwell's Eq'ns? For instance,

$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \partial^2 \vec{E}/\partial t^2$ has wave sol'ns, traveling at $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

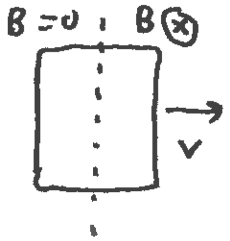
But, speed with respect to what? If you move to another inertial frame, does the speed of these waves change? Does the speed of these waves <u>depend</u> on the speed of the observer? Note that that <u>is</u> true of water or sound waves!! (In a speedboat, ocean waves move faster or slower with respect to <u>you</u> than with respect to the beach!)

Before Einstein, physicists largely thought this <u>must</u> be true of EM waves (including light) too: <u>obviously</u> speed of waves depends on $\vec{V}$ observer (!!) There should be a special frame (analogous to the rest frame of the water) where speed is $c = 1/\sqrt{\mu_0\epsilon_0}$ ... this is the "ether frame".

Michelson & Morley did a series of difficult but convincing exp'ts and found <u>no</u> <u>evidence</u> for any such rest frame, a "null result". There is no ether, no "medium" in which EM waves travel. It is a strange (bizarre, incredible!) <u>fact</u> that EM waves travel at the same speed c (in vacuum) <u>independent</u> of $\vec{V}$ observer (or <u>source</u>, for that matter)

3320 12-3

Back in ch. 7, Griffiths looked at a <u>wire</u> loop entering a region of $\vec{B}$



In the "magnet frame", the loop <u>moves</u>, + an EMF is generated by $\vec{F}_{mag} = q\vec{v} \times \vec{B}$ on charges in the wire. There is <u>no</u> $\vec{E}$ field anywhere! In the "frame moving with the loop", though, all charges in the wire are at <u>rest</u>, $\vec{F}_{mag} = 0$! Still, an EMF is generated, in this case by an $\vec{E}$ field created from Faraday's Law.

So the <u>causes</u> /sources of the EMF (& resulting physical current) seems <u>different</u> in the 2 frames, but the <u>law of physics</u> "$\mathcal{E} = -\frac{d\Phi}{dt}$" is the same for all observers. All of <u>Maxwell's Eq'ns</u> are "universal laws" in this way, true for any observers.

Einstein put this together: No ether, Maxwell's eq'ns are universal laws.

① The <u>laws</u> of physics apply in all inertial frames. (Maxwell's Eq'ns, in particular!)

② The speed of light $c = 1/\sqrt{\mu_0 \epsilon_0}$ is a "law of physics" too, all inertial observers will see EM waves travel at this <u>same</u> speed (in vacuum)

Item ② is hard to fathom. It leads to strange, radical consequences, about <u>basic</u> ideas of space, time, kinematics... But it's not a philosophical point, it's experimental fact! (A flashlight in a moving train makes a beam which travels at c with respect to <u>both</u> train <u>and</u> ground!)

Phys 3320 12-4

I'm going to 1st briefly summarize the "big 3" consequences of Einstein's 2 postulates. I'm hoping you've seen this in physics 3 (if not, you may want to spend a little extra effort with Griffiths!)

Definition: An "event" is something that happens at one place + time, (x, y, z, t)

E.g. a light beam strikes a detector, a firecracker explodes, ...

1) Relativity of simultaneity: In different inertial frames, two events at two different locations observed* as simultaneous in one frame can occur in a different time order in other frames (either time order, in fact)

(See next page for more explanation of what "observed" means in physics!)

2) Time dilation: Moving clocks are observed to run slow. The time between two events located at the same place in one frame (known as "the proper time") is always shorter than Δt observed for those same two events observed in any other (moving) inertial frame.

3) Length contraction: Moving objects are observed to be shorter in the direction of motion. The distance between the ends of an object at rest in one frame (the "proper length") is always longer than Δl measured by finding the distance between the ends as measured simultaneously in an other frame (moving parallel to the objects length)

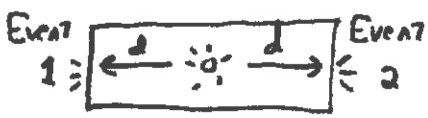
Phys 3320 12-5

Discussion: "observed" here does not mean "seen"! If you look (or take a picture) of a moving ruler, or clocks, you might see something quite different. (E.g, the length of a ruler moving towards you might look longer, not shorter!)

Observed means ... do it right ☺ Take into account light travel time. Start with a "grid" in your reference frame of rulers (all at rest) + clocks (at rest) at every point. Synchronize the clocks (using light & rulers), accounting for $\frac{d}{c}$ travel time.

If you look at a faraway clock, you will see an earlier time (picture it!) But if you observe that clock, it is in exact agreement with yours, in synch.

Simultaneity Example:



Car at rest. Flash a light at center. Events 1 & 2 are simultaneous with each other. (Light hits detector)

Same exp't as observed from a left-moving frame. that car above is apparently moving to right

In this frame, event 1 happens first, it is not simultaneous w. event 2. The speed of light is still c in this new frame (!) But the left wall moves forward to meet the beam, so detector #1 flashes at an earlier time than #2.

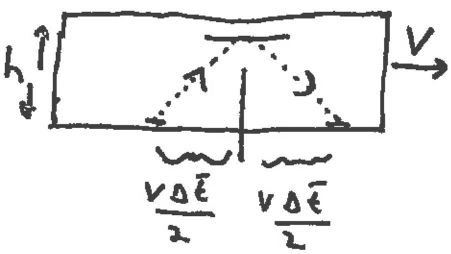
A third observer in a right-moving frame will observe even ② happening first.

So observers do not have to agree on whether distant events are simultaneous, it's a frame dependent concept!

3220 12-6

Time dilation example

Car is at rest. A light flashes, beams goes up h, hits a mirror, back down to start. Clock @ start reads $\Delta t = 2h/c$ (right?) This is called the proper time between the two events (light flash + light received). It's called proper time because we measured it with one stationary clock. That defines proper time!



Consider the same experiment observed in a left-moving frame. In this frame, distance traveled $= 2\sqrt{h^2 + \left(\frac{v\Delta \bar{t}}{2}\right)^2}$

Convince yourself from the picture!

$\Delta \bar{t}$ = time between 2 events = $\frac{\text{Total Distance}}{c \leftarrow \text{always!}} = \frac{2}{c}\sqrt{h^2 + \left(\frac{v\Delta \bar{t}}{2}\right)^2}$

Solving this eq'n, $c^2 \frac{\Delta \bar{t}^2}{4} = h^2 + \frac{v^2 \Delta \bar{t}^2}{4} \Rightarrow \frac{\Delta \bar{t}^2}{4}(c^2 - v^2) = h^2$

thus $\Delta \bar{t} = 2h/\sqrt{c^2 - v^2} = \frac{2h}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}} = \Delta t \cdot \gamma$

Where I define a "Lorentz factor" $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$

So $\Delta \bar{t}$ in a moving frame $> \Delta t_{proper}$, by a factor of γ, always.

Note that $\gamma \geq 1$

Non-relativistic motion $\Rightarrow \gamma \simeq 1$

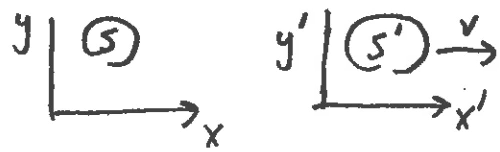
Ultra relativistic $\Rightarrow \gamma \rightarrow \infty$.

3320 12-7.

Lorentz Invariants: From this last example, different observers may disagree on time interval between events (next page, space intervals too!) But we don't disagree on everything. E.g., in that last example Δt was the shortest time possible between those 2 events. This shortest time, Δt_{proper} is something any observer can deduce. We all agree on it, it's a "Lorentz invariant" quantity. My $\Delta \bar{t} = \gamma \, \Delta t_{proper}$) We'll soon find more such invariants - they are useful & important.

Length contraction perpendicular to relative motion (not!)

I claim there won't be any funny business (i.e. disagreement among observers) regarding



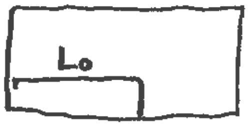
y & y' measurements.

Griffiths (p. 492) has a nice "physical proof" involving painting colored lines at height y & y' and comparing them. Read it & convince yourself!

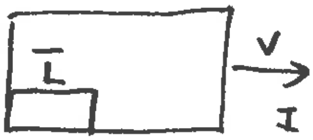
It amounts to saying we can compare $\perp$ directions in 2 frames directly without ever leaving our frames. There cannot be any difference, otherwise you could e.g. say that the frame with the absolute largest y is special. But no frame is special, all are equivalent!

3320 12-8

Length contraction parallel to motion



Car at rest, with ruler at rest. You measure $L_0 = \Delta X$ at one instant (say, $t=0$). This is the proper length, defined as the length of the object at rest. (Note that "at an instant" doesn't matter in this frame, since ΔX is same at all times)



Same ruler, observed from a (left) moving frame. I would need to measure $\bar{L} = \Delta \bar{X}$ by locating $\bar{X}_{right}$ & $\bar{X}_{left}$ in this frame at one instant in this frame. Since observers in other frames disagree on simultaneity, if I watch "rest guy" measure his ruler's end simultaneously to him, I'd say "Wait. You blew it, you measured the right end first, & I see the rule moving, so you got a distance that's not right!"

The result (I'll use a trick method next page, + more generally soon!) is

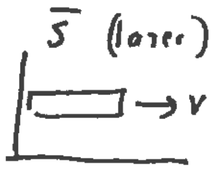
$\bar{L} = \frac{1}{\gamma} L_0$, the moving ruler is observed to be shorter than the rest ruler.

(This result is subtler than time dilation. It should be increasingly clear that we should stop with the "tricks", + get systematic! I really need a math toolbox to compare x's & t's in different frames. It's coming soon, that's the Lorentz transformations)

3320 12-9

"Trick" method: Watch the ruler move past me at speed V

$\bar{x}=0$, right end passes at $\bar{t}_1$



$\bar{x}=0$

Left end passes at $\bar{t}_2$

As always, $V *$ time = distance (in any given frame)

I conclude $\bar{L} = v \cdot \Delta\bar{t} = v(\bar{t}_2 - \bar{t}_1)$

I measured this with one clock, that means $\Delta\bar{t}$ is in fact the proper time between the two events "right end passes $\bar{x}=0$" + "left end passes $\bar{x}=0$"

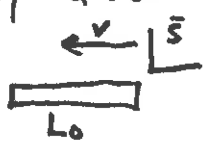
In another frame, say the ruler's rest frame, we know

$$\Delta t_{\text{those same 2 events}} = \underbrace{\gamma\, \Delta t_{\text{proper}}}_{\text{from notes, p.6!}} = \gamma\, \Delta\bar{t}$$

← convince yourself we got the γ factor on the correct side!

What does observer in rest frame of ruler see? A ruler, length L_0 at rest (!) and origin of $\bar{S}$ moving left, I measure 2 events at times t_1 (right end passes) and t_2 (left end passes). These 2 events are at opposite ends of a ruler at rest, a distance L_0 apart, a time Δt apart.

Picture it,



this point moves left at speed V in my new frame.

I observe $v \cdot \Delta t = L_0$ (same formula in my frame!). So $\boxed{\dfrac{L_0}{\Delta t} = v = \dfrac{\bar{L}}{\Delta\bar{t}}}$

But $\Delta\bar{t} = \frac{1}{\gamma}\Delta t$, so $\bar{L} = L_0 \dfrac{\Delta\bar{t}}{\Delta t} = \dfrac{1}{\gamma} L_0$.

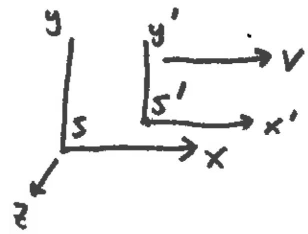
$\bar{L}$ is shorter, this is "Lorentz contraction"

3320 12-10

Lorentz TRANSFORMATIONS: the dictionary! Say an event happens at (x,y,z,t) in frame S, and the same event happens at (x',y',z',t') as observed in frame S'. We'll consider the simple case

① S' moves at speed V with respect to S

or, S moves at speed $-V$ with respect to S'!



• Here's the event

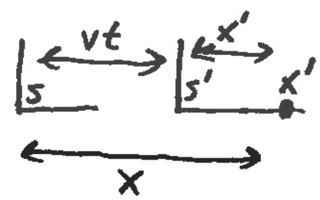
$x', \dots t'$ in S'

$x, \dots t$ in S.

② Lets let y, z axes match up: no funny business here!

③ Let's have $x = x' = 0$ when $t = t' = 0$ (Set your clocks to match when S' origin passes S origin)

Galileo (and all my common sense!) says an event at x' in S' will be at $x' + vt$ as seen in S



So
$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \;\equiv\; \text{or} \;\; \left\{\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}\right.$$

- Reasonable,
- common sense,
- agrees with Newton's laws

⋮

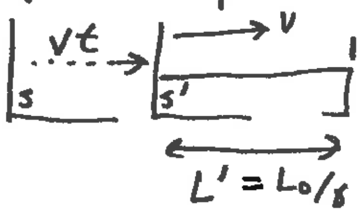
But, it's wrong! It does not agree with time dilation or length contraction or frame dependence of simultaneity!

3320 12-11

Our earlier results (contraction (L) + dilation (t)) help guide us...

Suppose a ruler (rest length L_0) sits (left end $x'=0$, right $x'=L_0$ in S')

What does S observe? A Lorentz contracted ruler, $L' = \frac{L_0}{\gamma}$, moving right at speed v.



An event at right edge of ruler will be at $x = vt + L'$ in frame S.

$$\underbrace{x}_{x \text{ in frame } S} = \underbrace{vt}_{\text{movement of origin in frame } S} + \underbrace{L'}_{\text{length of stick observed in frame } S}$$

So $x = vt + \frac{L_0}{\gamma} = vt + \frac{x'}{\gamma}$

→ In S', the stick is at rest! The right end is always at $x' = L_0$, independent of time!

So $x' = \gamma(x - vt)$

In words: position in moving frame $= \gamma \times$ (position in other frame minus $v \times$ time " " ")

What about t'? Here, I have a clever trick.

First, suppose S' moved left, not right.

Then of course $x = -vt + L' = -vt + x'/\gamma$

or $x' = \gamma(x + vt)$

In other words, if the frame moves left, just flip sign on v. (Easy!)

But, realize that as observed in frame S', it is frame S moving left with (same) relative speed v. So, flip your perspective, consider S' to be the "original frame", and S to be moving (with −v) we can write down the exact same results, this is the <u>principle</u> of relativity, no frame is special!

So I claim $x = \gamma(x' + vt')$ ← Convince yourself! I got this from previous page, $x' = \gamma(x - vt)$, by switch $x \leftrightarrow x'$, $t \leftrightarrow t'$, $v \leftrightarrow -v$

It's a key step! (γ is the <u>same</u> γ!)

But now look: If (1) $x' = \gamma(x - vt)$
(2) $x = \gamma(x' + vt')$ then plug (2) into (1)

$x' = \gamma(\gamma(x' + vt') - vt)$ and now <u>solve</u> this to get t

$x'(1-\gamma^2) - \gamma^2 v t' = -\gamma v t$, or $t = \gamma t' - \frac{(1-\gamma^2)}{\gamma v} x'$

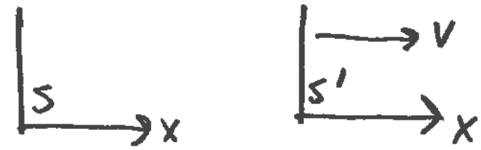
But note: $\frac{1-\gamma^2}{\gamma v} = \frac{\gamma(1-\gamma^2)}{\gamma^2 v} = \frac{\gamma}{v}\left(\frac{1}{\gamma^2} - 1\right) = \frac{\gamma}{v}\left(1 - \frac{v^2}{c^2} - 1\right)$
$= -\gamma v / c^2$

so $t = \gamma\left(t' + \frac{v}{c^2} x'\right)$

or, (flipping $x \leftrightarrow x'$, $t \leftrightarrow t'$, $v \leftrightarrow -v$) $t' = \gamma\left(t - \frac{v}{c^2} x\right)$

3320 12-13

The complete Lorentz transformations



$$x' = \gamma(x - vt) \qquad\qquad x = \gamma(x' + v't)$$

$$y' = y \qquad \text{or} \qquad y = y'$$

$$z' = z \qquad\qquad z = z'$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad\qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

with $\gamma = 1/\sqrt{1-v^2/c^2}$

- Newton's laws are <u>not</u> invariant under these rules. We must fix up Newton's laws, they are <u>wrong</u>!
- Maxwell's eq'ns <u>are</u> invariant under these rules. No need to fix them!
- These rules tell us how different observers measure events. You can use this to see how velocity (+ later acceleration) transforms. We'll do this next.
- These rules can be summarized in a matrix <u>notation</u> that will lead us to a powerful new notation & idea: 4 vectors. Coming soon
- These new notations will show us <u>invariants</u>, quantities which are <u>not</u> different in different frames. (Frame independence is a deep goal in physics, finding relationships independent of irrelevant details of local observers!)

3320 12-13 b: Lorentz transformations, again!

A summary, starting from scratch! 

What's the most general linear transformation (that overlaps at $t'=t=0$)?

$$\left.\begin{aligned} x' &= ax + bt \\ t' &= cx + dt \end{aligned}\right\}$$ this is as general as possible, assuming only linearity!

1) origin of x' ($x'=0$) moves at speed v in frame S, meaning that if $x'=0$, then $x = vt$. Plugging in, $x'=0 = a(vt) + bt \Rightarrow b = -av$

So $x' = a(x - vt)$ (Really, this is what we mean by "S' moves with velocity v"!)

2) Einstein ~~principle~~ postulate #1 says there is no preferred frame, in other words

$x = a(x' + vt')$ (Swapping $x \Leftrightarrow x'$, $t \Leftrightarrow t'$, $v \Leftrightarrow -v$ gives back the same physics, with same a out front!)

3) Einstein postulate #2 says speed of light is c in both frames.

So a light pulse emitted at origin follows $x = ct$ and $x' = ct'$!

Plugging in, $x' = a(x - vt) \Rightarrow ct' = a(ct - vt) = at(c-v)$

$x = a(x' + vt') \Rightarrow ct = a(ct' + vt') = at'(c+v)$

Multiplying these gives $c^2 tt' = a^2 tt'(c^2 - v^2)$

or $a^2 = 1/(1 - v^2/c^2)$.

So $x' = \dfrac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$ As before, from basic principles.

(Then, as on p.12, I follow exactly the same to finish the story)

Phys. 3320 12-14

Let's revisit time dilation + Lorentz contraction using our new formulas

Time dilation: If you have one clock in S, and two events happen at the location of that clock, $(\text{Event } 1)_{\text{in } S} = (x, t_1)$

$(\text{Event } 2)_{\text{in } S} = (x, t_2)$

so $(\Delta t)_{\text{in } S} = t_2 - t_1$.

In S', $\left.\begin{aligned} t_1' &= \gamma\left(t_1 - \frac{v}{c^2}x\right) \\ t_2' &= \gamma\left(t_2 - \frac{v}{c^2}x\right) \end{aligned}\right\}$ so $(\Delta t)_{\text{in } S'} = t_2' - t_1'$

$= \gamma(t_2 - t_1) = \gamma \Delta t_S$

This is time dilation, $(\Delta t)_{\text{in } S}$ is the minimum, it's the proper time

(one clock! Events happen at same $\underset{\Delta x = 0}{x}$). $\leftarrow$ (Note, $\Delta x' \neq 0$!)

Length contraction: If you have a stick at rest in S with length L_0 what does S' observe for length? You must pick one time in t' to measure both ends (that's what we mean by "observing length in S'")

So $(x_2)_{\text{in } S} = \gamma(x_2' + v t_2')$

$(x_1)_{\text{in } S} = \gamma(x_1' + v t_1')$

this is 0 by our definition!

subtract $(\Delta x)_{\text{in } S} = \gamma (\Delta x)_{\text{in } S'} + \gamma v (\Delta t')_{\text{in } S'}$

$\left((\Delta x)_{\text{in } S} \text{ is at rest, it's proper!} \right)$

So $\Delta x' = \frac{1}{\gamma} \Delta x_{\text{proper}}$, (as we had argued before from clocks + trains + trickery....

3320 12-15

Velocity Addition

Suppose I'm in frame S and an object moves past me at speed u

What that means is $u = dx/dt$ (in S)

What does an observer in S' measure for velocity of that same object?

Well, using $x' = \gamma(x - vt) \Rightarrow dx' = \gamma(dx - v\,dt)$

$t' = \gamma(t - \frac{v}{c^2}x) \Rightarrow dt' = \gamma(dt - \frac{v}{c^2}dx)$

with $\gamma = 1/\sqrt{1-v^2/c^2}$

So S' sees $\underbrace{u' = \frac{dx'}{dt'}}_{\text{usual def of speed!}} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - \frac{v}{c^2}dx)} = \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$

So $u' = \frac{u - v}{1 - \frac{uv}{c^2}}$ this is Einstein's subtraction rule :)

(If object moved with speed u' in frame S', I claim, by using the inverse relations, $(u)_{\text{in } S} = \frac{u'_{\text{in } S'} + V_{(S' \text{ with respect to } S)}}{1 + u'v/c^2}$

This is the more usual formula, the signs are simpler.

Notes: u is never greater than c (we'll see why soon)

- Adding velocities this way never exceeds c!
- Small velocities, denom is ≈ 1, + get intuitive $u = u' + v$.
- If u' or $v = c$, the result is still c, (constancy of c in all frames!!)

<u>Notation</u>: we can <u>summarize</u> Lorentz transformations with a powerful new notation. Usually, vectors in 3D ("3-vectors") have three components like $\vec{x}$ has $\begin{cases} x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$

(I'm writing the "component index" as a superscript for reasons to come)

The trick is to add a <u>fourth</u> component, ct, to the position vector. You could call it component #4 (some do!) but most physicists call it the "zero-th" component. So,

$x^0 \equiv ct$. And, since $t' = \gamma(t - \frac{v}{c^2}x)$ ← From Lorentz transformation on p. 13

$$\Rightarrow ct' = \gamma(ct - \frac{v}{c}x)$$

People abbreviate $\underline{\underline{\beta \equiv v/c}}$, and $\gamma = \underline{\underline{1/\sqrt{1-v^2/c^2}}} = 1/\sqrt{1-\beta^2}$

Thus:

$$\begin{array}{ll} ct' \leftrightarrow & x^{0\prime} = \gamma(x^0 - \beta x^1) \\ x' \leftrightarrow & x^{1\prime} = \gamma(x^1 - \beta x^0) \\ y' \leftrightarrow & x^{2\prime} = x^2 \\ z' \leftrightarrow & x^{3\prime} = x^3 \end{array}$$

← I used $x' = \gamma(x - vt)$
$= \gamma(x - \frac{v}{c} \cdot ct)$

This is called a transformation, or "boost", since we're shifting to a frame "boosted" by velocity $v\hat{x}$.

3320 12-17

Summarizing, $\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{call this matrix "}\Lambda\text{"}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

or more compactly, $(x')^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$

Here $(\text{Matrix})^\mu_\nu$ $\mu \to$ row element, $\nu \to$ column element.

["Einstein notation" is a shortcut: if I leave out the $\sum_{\nu=0}^{3}$ symbol, just assume it's there!]

- our superscript ("μ" or "ν") implies x' is itself a column vector.
- If we want row vectors, we'll lower the ~~super~~script $\Rightarrow$ subscripts

---

This notation reminds me a bit of rotations in 3-space.

- Any object which transforms under spatial rotations like $\vec{r}$ does (e.g momentum, $\vec{p}$) is a 3-vector.
- Any object which transforms under Lorentz boosts like x^μ does is called a "contravariant 4-vector," or simply "4-vector" (There are others!)

Example: Displacement $\Delta X \equiv X_A - X_B$. I claim this is also a 4-vector. I can prove it! $\Delta X^\mu = X_A^\mu - X_B^\mu$

If we boost frame, $(X_A')^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu X_A^\nu$ (or just $\Lambda^\mu{}_\nu X_A^\nu$ in Einstein form)

$$(X_B')^\mu = \sum_\nu \Lambda^\mu{}_\nu X_B^\nu$$

so $(\Delta X')^\mu = (X_A')^\mu - (X_B')^\mu = \sum_\nu \Lambda^\mu{}_\nu (X_A^\nu - X_B^\nu)$

meaning $$(\Delta X^\mu)' = \sum_\nu \Lambda^\mu{}_\nu \Delta X^\nu$$

That shows that $\Delta X^\bullet$ transforms exactly as X did!

Why do we care? We'll see shortly! (Not every collection of 4 things is a 4-vector!). First, one other notational trick:

Covariant 4-vectors: X_μ, or (X_0, X_1, X_2, X_3)

"Co" for "low", the index is a subscript now, + written as a "row"

we haven't defined it yet, it's still just notation

[with 3-vectors, $\vec{a} \cdot \vec{b} = \sum_i a_i b_i$. This defines the inner-product, & it's useful because if you rotate coordinates, a_i and b_i's all change, but $\vec{a} \cdot \vec{b}$ is invariant!]

3320 12-19

What would you expect for the "dot product" $a \cdot b$?

By analogy to 3 vectors, you might pick

$$a \cdot b \equiv \sum_{\nu=0}^{3} a_\nu b^\nu \quad \text{or simply} \quad a_\nu b^\nu \quad \text{in Einstein notation}$$

$$\text{or} \quad a_\mu b^\mu \quad \leftarrow \text{sum over any \underline{repeated} index!}$$

This is a definition - is it useful??

Consider $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$

In a boosted frame, $\bar{x}^\mu = \begin{pmatrix} c\bar{t} \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \bar{x^0} \\ \bar{x^1} \\ \bar{x^2} \\ \bar{x^3} \end{pmatrix} = \begin{pmatrix} \gamma x^0 - \gamma\beta x^1 \\ \gamma x^1 - \gamma\beta x^0 \\ x^2 \\ x^3 \end{pmatrix}$

Note that $\overline{x^{0}}^2 + \overline{x^1}^2 = (\gamma x^0 - \gamma\beta x^1)^2 + (\gamma x^1 - \gamma\beta x^0)^2$

$$= \left(\gamma^2 x^{0^2} - 2\gamma^2\beta x^0 x^1 + \gamma^2\beta^2 x^{1^2}\right) + \left(\gamma^2 x^{1^2} - 2\gamma^2\beta x^0 x^1 + \gamma^2\beta^2 x^{0^2}\right)$$

This is a big mess. But, if we stuck a "$-$" sign on the 1st term, by hand, we'd get a lot of beautiful cancellations:

$$-\overline{x^{0}}^2 + \overline{x^1}^2 = \gamma^2 x^{0^2}(-1+\beta^2) + \gamma^2 x^{1^2}(-\beta^2+1) \quad \text{(the } x^0 x^1 \text{ terms cancelled)}$$

$\uparrow$

!!

3320 12-20

It gets better: $\gamma^2 = \frac{1}{1-\beta^2}$ so $\gamma^2(1-\beta^2) = 1$, so

$$-\overline{x}^{0\,2} + \overline{x}^{1\,2} = -x^{0\,2} + x^{1\,2}$$

So check this out: If we <u>decide</u> (by "fiat") to define

$x_0 = -x^0 = -ct$, then our dot product $a \cdot b = \sum_{\mu=0}^{3} a_\mu b^\mu$

stick this in by hand!

gets that "−" sign introduced on the $x_0 x^0$ term, and

$$x \cdot x \equiv x_\mu x^\mu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

$$\overline{x} \cdot \overline{x} = \overline{x}_\mu \overline{x}^\mu = -(\overline{x}^0)^2 + (\overline{x}^1)^2 + (\overline{x}^2)^2 + (\overline{x}^3)^2$$

But we just showed $-\overline{x}^{0\,2} + \overline{x}^{1\,2} = -x^{0\,2} + x^{1\,2}$, and $x^2 = \overline{x}^2$
$x^3 = \overline{x}^3$

so $x \cdot x$ is <u>invariant</u>, you get the same answer in <u>any</u> reference frame.

(In general, it's easy enough to show that

$a \cdot b \equiv a_\mu b^\mu$ is invariant. If you boost frames + compute $\overline{a}\,\overline{b}$ you get the same result)

3320 12-21

Summary: <u>Covariant</u> 4-vector $x_\mu \equiv (x_0, x_1, x_2, x_3)$

$= (-x^0, x^1, x^2, x^3)$

$= (-ct, x, y, z)$

[Remember, $x_0 = -x^0$, so

$\sum_\mu x_\mu x^\mu = -c^2t^2 + x^2 + y^2 + z^2$ ↑ !] → Rigged up, so $a_\mu b^\mu$ is "Lorentz invariant"

<u>Example</u>: flick on a bulb at $x=y=z=t=0$ in frame S.

I expect a "wave front" is at $x^2+y^2+z^2 = c^2t^2$ (do you agree?)

What is measured in $\bar{S}$? Here

$$\bar{x} = \gamma(x - vt)$$
$$\bar{y} = y$$
$$\bar{z} = z$$
$$\bar{t} = \gamma(t - vx/c^2)$$

I claim $\bar{x}^2 + \bar{y}^2 + \bar{z}^2 = c^2\bar{t}^2$ too. (you can do the algebra, but we really just did it on the previous page!)

So the formula "$x_\mu \cdot x^\mu = 0$" describes the wavefront, in <u>any</u> frame!

The formula is written in a <u>manifestly</u> covariant ("frame independent") way. x, and t are <u>not</u> frame independent, but the formula/relationship describing the wave front is!

3320 12-22

<u>Example #2</u>: Consider Δx^μ (which we proved is a 4-vector!)

$$\Delta x_\mu \Delta x^\mu = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$= \underbrace{d^2}_{\text{spatial separation of 2 events}} - c^2 \underbrace{\Delta t^2}_{\text{time separation of those same 2 events}}$$

Neither d nor Δt is "observer independent"! E.g, what is simultaneous in <u>one</u> frame, $\Delta t = 0$, need <u>not</u> be in another. But, this <u>combo</u> is.

So Defining I = "invariant interval" $\equiv \Delta x_\mu \Delta x^\mu$

the <u>formula</u> is <u>manifestly</u> <u>Lorentz</u> <u>Invariant</u>, (by the mathematical form, $a_\mu b^\mu$)

I is the <u>same</u>, measured by any inertial observer. (It's also called the "space-time interval.") Any time you can write physics formulas in manifestly invariant combinations, you know you're on the right track, since you're describing observer independent things.

We'll come back to this. But, I itself is interest + important (like most invariants!) Let's investigate it a bit...

3320 12-23

— Consider 2 <u>events</u> at one location in some frame (e.g., you ring a doorbell, wait, then ring it again)

$I_{\text{these 2 events}} = d^2 - c^2 \Delta t^2 = 0 - c^2 \Delta t^2 < 0$.

That means <u>any</u> observer (in <u>any</u> frame) <u>must</u> agree $I < 0$.

This is called "timelike separated events" (for obvious reasons)

- <u>No</u> observer could claim these events are simul<u>taneous</u>, because then $I = d^2 - c^2 \Delta t^2$ would be $d^2 (> 0)$, impossible. (One event is definitely <u>first</u>)

— Consider 2 events, simul<u>taneous</u> but different locations, in some frame (e.g 2 doors at 2 houses being rung simultaneously in the "houses frame")

$I_{\text{these 2 events}} = d^2 - c^2 \Delta t^2 = d^2 - 0 > 0$

So any observer (any frame) <u>must</u> agree $I > 0$

This is called "spacelike separated events". (<u>No</u> observer will claim these occur at the same place!)

— If $I = 0$, $d^2 = c^2 \Delta t^2$, this is called "lightlike" separation, since a light beam could go from one event to the other.

3320 12-24

Note: If two events A & B are <u>timelike</u>, $I_{AB} < 0$,

I claim there <u>does</u> exist a frame in which $d = 0$!

Suppose in frame S they are <u>not</u> at the same place, $\Delta x \neq 0$

In $\bar{S}$, $\Delta \bar{x} = \gamma(\Delta x - v\Delta t)$

If I <u>choose</u> $v = \dfrac{\Delta x}{\Delta t}$, then $\Delta \bar{x} = 0$.

I know this is possible because $I_{AB} = \Delta x^2 - c^2 \Delta t^2 < 0$

$\Rightarrow \Delta x < c\Delta t$

$\Rightarrow v = \dfrac{\Delta x}{\Delta t} < c$. No problem!

But, with <u>spacelike</u> events (~~$\Delta x \neq 0$, Δt ... same~~)

there is <u>no</u> frame in which $\Delta \bar{x} = 0$. Why not?

Originally, $\Delta x^2 - c^2 \Delta t^2 > 0$, so $\Delta x > c\Delta t$

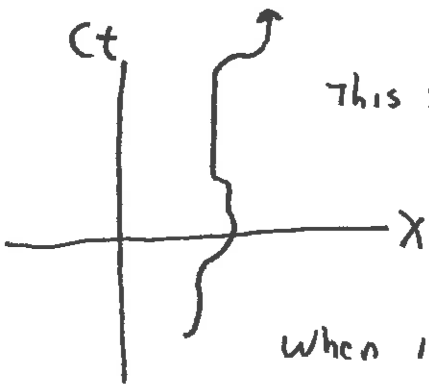
Thus, $\Delta \bar{x} = \gamma(\Delta x - v\Delta t)$, but no frame can have $v > c$

(that gives imaginary γ, and all kinds of trouble!)

If $\Delta \bar{x} = 0 \Rightarrow v = \Delta x / \Delta t > c$, no go!

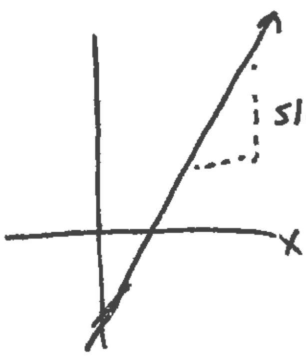
3320 12-25

One way to think about intervals is with "space-time diagrams", or "Minkowski" plots. We simplify to 1-D space, + plot in an unusual way, with ct on the vertical axis:



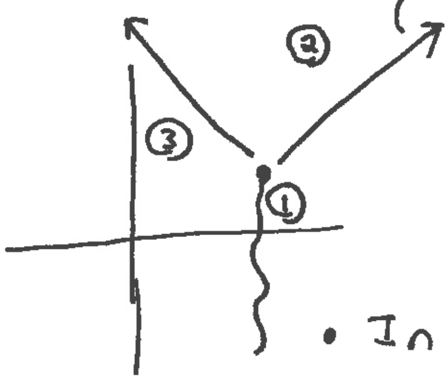
This is a "world line", a plot of x vs t (or technically ct vs x) of some object as it moves through space & time.

When it's vertical, it's at rest, $\Delta x = 0$ there!



slope $= \frac{c\Delta t}{\Delta x} = 1/\beta$

So no slope is ever less than 1 on these plots, (since $\beta \le 1$)



slope = 1, this is called the "light cone" (It would be a cone if we added "y" in!)

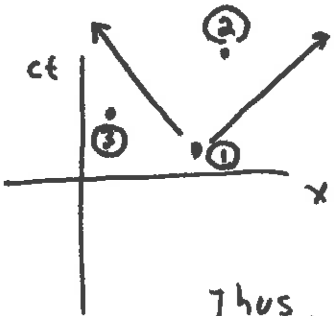
A particle can only communicate inside its forward light cone.

- In this figure, event ② could be caused by ①

But event ③ could not be caused by event ①

Why? Let's investigate a bit more...

3320 12-26



$I_{21} < 0$, (2) and (1) are time like

$\Delta x_{12}^2 < c^2 \Delta t_{12}^2$ if 2 is inside the forward light cone.

Thus, no frame is possible which finds $\Delta t_{12} = 0$

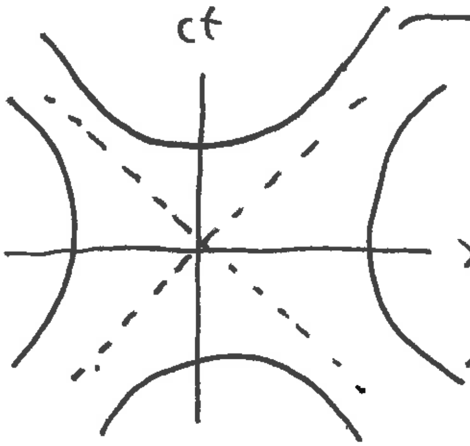
In fact, in the frame where $\Delta x_{12} = 0$, Δt_{12} is a minimum, (because it's the proper time!) You can, thus, never reverse the time order of these events: cause precedes effect for all observers.

$I_{13} > 0$, (1) and (3) are space like. $\Delta x_{13}^2 > c^2 \Delta t_{13}^2$

you can reverse the time order, Δt_{13} can be 0, +, or −, but that's ok, there's no violation of causality, because (1) can't cause (3) (or vice-versa), they're just "elsewhere".

Light signals are the fastest possible "causes", yielding $\Delta t = \frac{\Delta x}{c}$, and Δx_{13} is bigger than this!

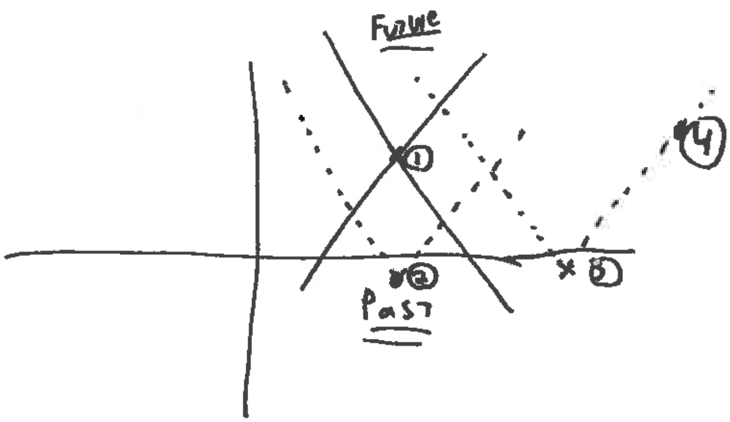
3320 12-27



These events all have a common timelike "I" w.r.t. the origin, because $I = -c^2t^2 + \Delta x^2$ defines a hyperbole!

These events have common spacelike I

- Lorentz boosts can only move you to a different point on the same hyperbole (keeping I invariant!)
(And, you cannot boost from back<u>ward</u> light-cone to for<u>ward</u>)



Event 2 could be causally related to 1, but not 3

4 is "on the light cone" of 3

To a photon, $I = 0$ between events, (so perhaps <u>all</u> events it "experiences" are at the same time??)

[See next page]

3320 12-28

Invariants: proper time

consider a particle in my reference frame moving with $\vec{u} = \frac{dx}{dt}$ (w.r.t. me)

In the particles rest frame (call that S'), it would measure

proper time $d\tau = \frac{1}{\gamma} dt$ where $\gamma = 1/\sqrt{1-u^2/c^2}$

Note: $c\,d\tau = c\sqrt{1-u^2/c^2}\,dt = \sqrt{c^2dt^2 - u^2dt^2} = \sqrt{c^2dt^2 - \left(\frac{dx}{dt}\right)^2 dt^2}$

$= \sqrt{c^2dt^2 - dx^2} = \sqrt{-dx_\mu dx^\mu}$, (with dx^μ = displacement of particle)

- This last expression is "manifestly invariant", you can re-express dx_μ in any inertial frame, + you get the same result.

- $d\tau$ = "proper time", it's invariant, all observers agree on it

- It is the time interval "experienced" in the rest frame of the particle itself. In that frame, $u = 0$, and so $dt = d\tau$.

In general, if you have a 4-vector and multiply it by an invariant, you still have a 4-vector.

This turns out to be useful! →

3320 12-29

Consider $\eta^\mu \equiv \dfrac{dx^\mu}{d\tau}$ $\longrightarrow$ a 4-vector, the "4-displacement" of a particle
$\longrightarrow$ an invariant, proper time " " " "

So this is itself a 4-vector, meaning we know exactly how it transforms

Use the Lorentz transformations, $\bar{\eta}^\mu = \Lambda^\mu_\nu \eta^\nu$, we can figure out $\bar{\eta}$ in any frame just as easily as we can transform x and t.

Units suggest we call this a sort of "4-velocity" (m/s)

$\to$ see previous page

Note: $\eta^0 = \dfrac{dx^0}{d\tau} = c\dfrac{dt}{d\tau} = c\gamma = c/\sqrt{1-u^2/c^2}$

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \vec{u}\gamma = \vec{u}/\sqrt{1-u^2/c^2}$$

$\hookrightarrow$ this is called "proper velocity" $\hookrightarrow$ this is "normal velocity"

$\vec{\eta}$ is a slightly curious beast by itself, $\vec{\eta} = \dfrac{d\vec{x}}{d\tau}$ $\to$ displacement in S
$\to$ proper time, measure in particle's rest frame

Sort of a "hybrid"

But, η^μ is well-defined & useful. Remember how 4-vector transform:

$$\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1)$$
$$\bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^0)$$

(where $\bar{S}$ moves with speed "βc" in the x-direction in frame S)

Phys 3320 12-30

- Note that transforming regular velocity $d\vec{x}/dt$ is nastier, because $\vec{x}$ transforms, but so does dt!

- For any 4-vector, its square is invariant:

Here, $\eta^2 \equiv \eta \cdot \eta \equiv \eta_\mu \eta^\mu = -\eta^{0\,2} + \vec{\eta}^{\,2}$

$$= \frac{-c^2}{1-u^2/c^2} + \frac{u^2}{1-u^2/c^2} = \frac{u^2-c^2}{(c^2-u^2)/c^2} = -c^2$$

so $\eta^2 = -c^2$, indeed very obviously invariant! (True for any moving object)

- η^μ is itself the 1st step in constructing more 4-vectors. You might consider one more derivative (to get "4-acceleration", perhaps leading to "4-Force"). We'll get there, but first, consider

Four momentum: $p^\mu \equiv m\eta^\mu = m \dfrac{dx^\mu}{d\tau}$, (it's mass * 4-velocity)

Here $m \equiv$ rest mass of the object, mass measured in the rest-frame of the object. This is manifestly invariant, so p^μ must be a 4-vector.

$\vec{p} = \dfrac{m}{\sqrt{1-u^2/c^2}}\vec{u}$ The relativistic momentum is not $m\vec{u}$ (there's this extra γ in there!)

$p^0 = m\eta^0 = mc\gamma = mc/\sqrt{1-u^2/c^2}$ what's this? We'll see, next page...

3320 12-31

<u>Momentum</u> : In rest frame of the particle, $\vec{u}=0$, and

$\vec{p}=0$
$p^0 = mc$, or $p^{\mu}_{\text{rest frame}} = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$

In <u>my</u> frame S (where particle moves with velocity $\vec{u}$), p^0 is <u>bigger</u>.

And, the faster it goes, the bigger p^0 gets. It's a scalar object that increases with speed... sounds a bit like an energy...

To get units right, let's <u>define</u> a quantity

$$\underline{\underline{E \equiv cp^0}} = \underline{\underline{\gamma mc^2}} \quad \text{"Relativistic energy"}$$

In <u>rest</u> frame, $\underline{\underline{E = mc^2}}$ "rest energy"

$\underline{\underline{E - E_{rest}}} = (\gamma - 1) mc^2$ is "Kinetic energy" (the <u>part</u> of the energy due just to motion)

– Why these classical names for these arbitrary-looking definitions? Units are a clue, but the "non-relativistic limit" is the <u>real</u> reason.

If $u/c \ll 1$, then $\gamma = \frac{1}{\sqrt{1-u^2/c^2}} = (1-u^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots$

Let's use this to examine the kinetic energy...

$$E_{kin} = (\gamma - 1) mc^2 \underset{\text{non-rel limit}}{=} \left(\tfrac{1}{2} u^2/c^2\right) \cdot mc^2 = \tfrac{1}{2} m u^2 \quad \text{Hey!}$$

<u>Exactly</u> our usual old kinetic energy... hence the name!

For very relativistic particles, $u/c \approx 1$, γ is big

$$KE = mc^2(\gamma - 1) \approx \gamma mc^2 = E$$

and $|\vec{p}| = \gamma m u \approx \gamma m c$

so $|\vec{p}| \approx E/c$ for very relativistic particles, $(|\vec{p}| >> mc)$

Note, in this case $p \neq mv$ at all!

$KE \neq \frac{1}{2}mv^2$ at all!

What about the invariant $p_\mu p^\mu$? Work it out!

$$p_\mu p^\mu = -p^{0\,2} + \vec{p}^{\,2} = \gamma^2(-m^2c^2 + m^2u^2) = \frac{1}{1-u^2/c^2} \cdot m^2(u^2 - c^2) = -m^2c^2$$

- This tells you a quick definition/means to find the rest mass.
- This shows us visibly that rest mass is indeed manifestly invariant
- We can eliminate u here, so get

$$-p^{0\,2} + \vec{p}^{\,2} = -m^2c^2 \Rightarrow -\frac{E^2}{c^2} + \vec{p}^{\,2} = -m^2c^2$$

or $\boxed{E^2 = \vec{p}^{\,2}c^2 + m^2c^4}$ very handy! The "relativistic version" of $E = p^2/2m$

Full energy, $E = \gamma mc^2$, (it's not E_{Kin} which is $E - mc^2$)

[When we change frames, $\bar{p}^\mu = \Lambda_\nu{}^\mu p^\nu$ mixes up $\vec{p}$ & E, like space & time. But $p_\mu p^\mu$ is always invariant!]

Discussion

- $|\vec{p}|$ and E grow (indefinitely!) with speed. But, it's not the old non-relativistic "p is linear with v" and "E is quadratic in v" any more!

- Experimentally, p^μ is conserved ← this is in fact the true reason why we defined p^μ as we chose to!

This means $\sum \vec{P}_{\text{all particles}}$ is conserved in interactions
$\sum E_{\text{all particles}}$ " " "

"Conserved" is not "invariant"! Invariance refers to something that is the same for all (different) inertial frames/observers

Conserved refers to something that any observer says does not evolve or change with time.

- So, e.g. E is conserved, but is not invariant (it's one component of a 4-vector, + is different in different frames!)

- m is invariant, but it not conserved (!) you can have interactions where m changes, e.g when matter + antimatter annihilate to give only photons which have zero rest mass!)

- Charge is invariant & conserved!

phys 3320, 12-34

Relativistic collisions : 4-vectors can help simplify the story, look for invariants!

Ex: Antiprotons were discovered at the Berkeley Bevatron,

$p + p \rightarrow p + p + p + \bar{p}$ (Energetic protons hit protons at rest, yielding what you started with and $p\,\bar{p}$ pairs, producing particles from energy!)

Initial (Lab frame) $\longleftrightarrow$ Final (Lab frame)

[What energy was needed for the initial p? This was the driving consideration for the lab designers!]

Initial (Center of momentum frame) $\longleftrightarrow$ Final cm frame.

[At "threshold", in this frame, all 4 final particles will just sit there.]

$p_\mu^{TOT} p^\mu_{TOT}$ is frame invariant (same in lab and CM), but also

p_μ^{TOT} is conserved (same before + after the collision)

Look at the Lab picture: $P^\mu_{TOT,\,init} = \underbrace{(\gamma_L mc,\ \vec{P}_{Lab})}_{\text{incoming proton}} + \underbrace{(mc,\ 0)}_{\text{target proton}}$

In the CM picture: $P^\mu_{TOT,\,final} = (\underbrace{4mc}_{\text{4 objects at rest.}},\ 0)$ (Remember $p^0 = E/c$)

phys 3320, 12-35

By invariance & conservation together

$$\underbrace{p_{\mu\,\text{tot, init}}\; p^{\mu}_{\text{tot, init}}}_{\text{in lab frame}} = \underbrace{p_{\mu\,\text{tot, final}}\; p^{\mu}_{\text{tot, final}}}_{\text{in CM frame}}$$

so $\vec{P}_{Lab}^{\,2} - m^2c^2(1+\gamma_L)^2 = 0 - (4mc)^2$

[But also note, for the incoming proton, $E_{Lab} = \gamma_{Lab}\, mc^2$ ← always

and $E_{lab}^2 = \vec{P}_{lab}^{\,2} c^2 + m^2c^4$ ← always

so $\vec{P}_{Lab}^{\,2} = \frac{E_L^2}{c^2} - m^2c^2 = \frac{(\gamma_L mc^2)^2}{c^2} - m^2c^2 = m^2c^2(\gamma_L^2 - 1)$]

Put this in the formula above

$$m^2c^2(\gamma_L^2-1) - m^2c^2(1+\gamma_L)^2 = -16\, m^2c^2$$

so $-1 - 1 - 2\gamma_L = -16$ or $2\gamma_L = 14$ or $\gamma_L = 7$

This says you need $E_{kin,\,lab} = (\gamma_L - 1)\, mc^2 \simeq 6\, mc^2$

About <u>6 GeV</u>

They built the Bevatron ("Beva" for "Billion eV"!) with <u>just</u> over this energy, + got the Nobel prize for discovering the first antiprotons!

[See Griffiths for more fun examples of using invariants to solve collision problems!]

Let's return to 4-vectors & consider "4-force".

- Non-relativistically, $\vec{F}_{net} = d\vec{p}/dt$. Turns out this is relativistically correct if you use $\vec{p} = \gamma m \vec{v}$. Nice! However, since that's d/dt, not $d/d\tau$, this equation does not transform frames nicely, it's not "covariant"
- you can define a proper 4-vector, the "Minkoswki force",

$$K^\mu = \frac{dp^\mu}{d\tau} \rightarrow \text{note, } d\tau \text{ is invariant} \quad \Bigg\} \text{ Manifestly a 4-vector}$$

so $$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \gamma \vec{F}$$

$$K^0 = \frac{1}{c}\frac{dE}{d\tau} \quad \left(= \frac{1}{c}\, c\, \vec{F}\cdot\vec{v} \text{ as we'll show below}\right)$$

→ K^μ is formally important, but I've never <u>used</u> it to calculate motion. In a given frame, $\vec{F} = d\vec{p}/dt$ is the way to go! In E+M, e.g., $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = d\vec{p}/dt$ is <u>correct</u>, with $\vec{p} = \gamma m \vec{v}$.

Note: $E^2 = p^2c^2 + m^2c^4$, so $2E\, dE/dt = 2\vec{p}\cdot\frac{d\vec{p}}{dt}c^2 + 0$

$$= 2\vec{p}\cdot\vec{F}\,c^2$$

Recall $$\left.\begin{aligned}\vec{p} &= \gamma m \vec{v}\\ E &= \gamma m c^2\end{aligned}\right\} \text{so } \frac{\vec{p}c^2}{E} = \vec{v}$$

<u>Combining</u>, we get $dE/dt = \vec{F}\cdot\vec{v} = \vec{F}\cdot d\vec{x}/dt$

so $dE = \vec{F}\cdot d\vec{x}$, this is the same old work-energy theorem we know + love in non-relativistic physics!

3320 12-37

EM fields in special relativity

Consider $K^\mu = \frac{dp^\mu}{d\tau} = \left(\gamma \frac{\vec{F}\cdot\vec{v}}{c}, \gamma \vec{F}\right)$ from previous page

with $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$, we have $K^\mu = \left(\gamma q \frac{\vec{E}\cdot\vec{v}}{c}, \gamma q(\vec{E}+\vec{v}\times\vec{B})\right)$

Since we know K^μ is a 4 vector, we know how its components transform

In principle, you can use this to deduce how $\vec{E}$ + $\vec{B}$ transform.

($\vec{E}$ is not itself part of a simple 4-vector! Neither is $\vec{B}$.)

Instead, what we will find is a trick due to Minkowski:

$$K^\mu = \underbrace{q\,\eta_\nu F^{\mu\nu}}_{\text{summed over } \eta\text{, as always}}$$

with

$$\begin{array}{c} F^{\mu\nu} \\ \uparrow\ \uparrow \\ \text{row}\ \text{column} \end{array} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

[$F^{\mu\nu}$ is a tensor. This is an object which transforms like

$$\bar{F}^{\mu\nu} = \Lambda_\gamma{}^\mu \, F^{\gamma\delta} \, \Lambda_\delta{}^\nu$$]

It's antisymmetric, $F^{\mu\nu} = -F^{\nu\mu}$

This tells us how E & B transform. It's not as simple as Lorentz transformations... but almost! We'll come back to it soon! (See next page)

E.g. $K^0 = q[\eta_0 F^{00} + \eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}] = q[0 + \eta_x E_x/c + \cdots]$

$= q\vec{\eta}\cdot\vec{E}/c = q(\gamma\vec{v}\cdot\vec{E}/c)$ as claimed above...

3320 12-38

Just so you have it, here's the <u>result</u> for how E & B transform

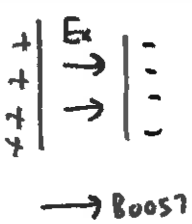
Griffiths derives these all by <u>hand</u>, but there are many ways to get them, (including using the Λ's + Minkowski tensor of previous page)

$$t' = \gamma\,(t - vx/c^2)$$
$$x' = \gamma\,(x - vt)$$
$$y' = y$$
$$z' = z$$

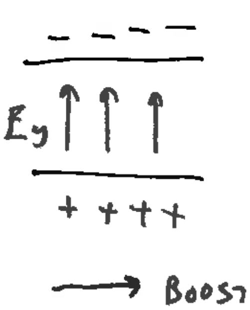
<u>Usual</u> <u>Lorentz</u> eq'ns

$$E_x' = E_x$$
$$E_y' = \gamma\,(E_y - v B_z)$$
$$E_z' = \gamma\,(E_z + v B_y)$$
$$B_x' = B_x$$
$$B_y' = \gamma\,(B_y + v/c^2\, E_z)$$
$$B_z' = \gamma\,(B_z - v/c^2\, E_y)$$

<u>Not</u> the usual Lorentz eq'ns, but perhaps simpler than you might have guessed. This arises directly from relativity alone, see prev. page, or below...



If you boost this configuration in +x direction, the plate spacing Lorentz contracts, but so what? E is independent of Δx! So E_x does <u>not</u> transform in an x-boost



But if you boost this, Δx shrinks, the charge density σ on the plates grows by γ, so $E_y' = \gamma E_y$, see formulas above. See Griffiths for explanations of the rest!

Phys 3320 12-39

Let's step back, I got a little ahead of myself!

Consider charges + currents in different frames.

- Charge is conserved <u>and</u> a Lorentz invariant quantity.

(These are experimental facts) But charge <u>densities</u> and <u>currents</u> depend on derivatives which make them frame dependent.

- Define ρ_0 = "proper charge density" $= \dfrac{Q}{V}$ in charges' <u>rest frame</u>
(invariant!)

Then $J^\mu \equiv \rho_0 \eta^\mu$ is a 4-vector.

Recall $\eta^\mu = \gamma(c, \vec{u})$, so $J^\mu \equiv (c\rho, \vec{J}) = \gamma(c\rho_0, \vec{u}\rho_0)$

Thus, $\rho = \gamma\rho_0$, which seems reasonable. The charge density gets dilated by γ, due to the <u>contraction</u> of volume in $\dfrac{Q \rightarrow \text{invariant}}{V \rightarrow \text{contracts}}$

Note $\vec{J} = \rho\vec{u}$, just as we have <u>always</u> said before!

We have always written charge conservation as

$\vec{\nabla}\cdot\vec{J} + \partial\rho/\partial t = 0$, This is still true in any frame.

Let's rewrite it with 4-vector notation!

3320 12-40

$$\frac{\partial J^0}{\partial x^0} = \frac{\partial (c\rho)}{\partial (ct)} = \partial \rho / \partial t$$

and $\frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} = \vec{\nabla} \cdot \vec{J}$, so $\vec{\nabla} \cdot \vec{J} - \frac{\partial \rho}{\partial t} = 0$

can be written as $\sum_{\mu} \frac{\partial J^\mu}{\partial x^\mu} = 0$, or in Einstein notation $\partial_\mu J^\mu = 0$

with ∂_μ shorthand for $\frac{\partial}{\partial x^\mu}$

This is not just elegant + compact, when I look at the formula $\partial_\mu J^\mu = 0$, it is manifestly invariant. (It makes it immediately apparent that charge conservation is true in all frames, + since $J^\mu = \rho_0 \eta^\mu$, suggests that proper charge density must be a Lorentz invariant.)

+ thus charge itself is Lorentz invariant.

Now go back to that $F^{\mu\nu}$ tensor we introduced. (p. 37)
Here's another independent way to come up with it: I claim

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$ is a shorthand for writing down

$$\left[\begin{array}{l} \nabla\cdot\vec{E} = \rho/\varepsilon_0 \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \end{array}\right.$$

$\leftarrow$ this is just the $\mu=0$ part of the eq'n
$\leftarrow$ " " " the rest of the eq'n.

(I'll show one of these below) The point is, we now have an elegant + compact way to write Maxwell's eq'ns in "covariant form". It's manifest that they hold in all frames, written this way.

Example: $\frac{\partial}{\partial x_\nu} F^{\mu\nu} = \mu_0 J^\mu$. Let's consider $\mu = 0$
Remember, L.H.S. is summed $\sum_{\nu=0}^{3}$

Go back to page 37 for the definition of $F^{\mu\nu}$. With $\mu=0$, we are looking at the top row of $F^{\mu\nu}$ which is $0 \quad \frac{E_x}{c} \quad \frac{E_y}{c} \quad \frac{E_z}{c}$

so $\mu_0 J^0 = 0 + \frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$, in other words

$\mu_0 c\rho = \frac{1}{c}\vec{\nabla}\cdot\vec{E}$, or $\vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \rho/\varepsilon_0$

There it is!

We can define one more tensor, $G^{\mu\nu} \equiv \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$

Again, antisymmetric

[This too (it can, and must be shown!) is a tensor, it transforms as a tensor
$\bar{G}^{\mu\nu} = \Lambda^{\mu}_{\lambda}\, G^{\lambda\sigma}\, \Lambda^{\nu}_{\sigma}$]

Then I claim $\partial_\nu G^{\mu\nu} = 0$ is shorthand for the other two maxwell's eq'n: $\mu = 0$ gives $\vec{\nabla}\cdot\vec{B} = 0$
$\mu \neq 0$ " $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$] check it!

So, to summarize, we have rewritten Maxwell's eq'ns in manifestly covariant way:

$$\left.\begin{aligned} \partial_\nu F^{\mu\nu} &= \mu_0 J^\mu \\ \partial_\nu G^{\mu\nu} &= 0 \end{aligned}\right]$$

These eq'ns are frame independent, these are the deep statements about nature which are true even when you Lorentz boost or rotate...

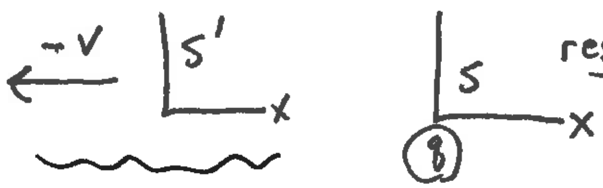
and $K^\mu = q\,\eta_\nu F^{\mu\nu}$ is the "force-formula"

All of E&M, written compactly, manifestly respecting the constraints of relativity!

3320 12-43

Let's end with an example, that shows some of the power of these new (fierce!) notations + ideas.

Consider 1st a point charge at rest. We know the field, a simple coulomb $\vec{E}$ field, nothing more. Now I ask, what is the EM field of a moving charge? We skipped this in Ch. 10, but with relativity it's easy enough!



rest frame of charge

This is the frame in which the charge moves with $+V$.

In S, $\begin{cases} \vec{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\vec{r}}{r^3} = \dfrac{q}{4\pi\epsilon_0}\dfrac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \\ \vec{B} = 0 \end{cases}$

In S', we want to know $E' + B'$, that's the field of a moving charge.

[From p. 38] (with $V \to -V$) see picture above!

$$\left.\begin{aligned} E_x' &= E_x \\ E_y' &= \gamma(E_y + vB_z) \\ E_z' &= \gamma(E_z - vB_y) \\ B_x' &= B_x \\ B_y' &= \gamma(B_y + V/c^2\, E_z) \\ B_z' &= \gamma(B_z + V/c^2\, E_y) \end{aligned}\right]$$

and also, as always (again $v \to -v$)

$$\left[\begin{aligned} x &= \gamma(x' - vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' - \frac{V}{c^2}x'\right) \end{aligned}\right.$$

So $E_x' = E_x = \frac{q}{4\pi\varepsilon_0} \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{q\,\gamma(x'-vt')}{4\pi\varepsilon_0 \left(\gamma^2(x'-vt')^2 + y'^2 + z'^2\right)^{3/2}}$

[Note: Even though $E_x' = E_x$, the coordinate transformation into our new frame introduces γ and the t' dependence!

Next, $E_y' = \gamma E_y + 0 = \frac{q\,\gamma\, y'}{4\pi\varepsilon_0 (\text{ same denom })^{3/2}}$

similar for E_z'

[Note: Even though E_y' formula has an explicit γ in it, the Lorentz coordinate transformation does not, so in the end all 3 components of E' transform very similarly, with the same γ]

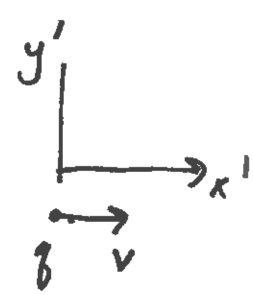
So $E' = \gamma \frac{q}{4\pi\varepsilon_0} (x'-vt', y', z') / (\gamma^2(x'-vt')^2 + y'^2 + z'^2)^{3/2}$

At $t'=0$, when q passes my origin,

$E' = \gamma q/4\pi\varepsilon_0 \; (x', y', z') / (\gamma^2 x'^2 + y'^2 + z'^2)^{3/2}$

Go to $z'=0$, i.e.

"in the plane of the page)



3320 12-45

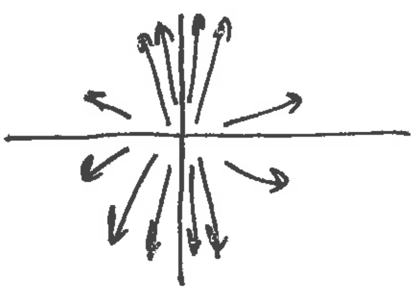
(with $z'=0$)

Along the $y'=0$ line (this is the x' axis!) $\quad E'_{(x',0,0)} = \frac{\gamma q}{4\pi\varepsilon_0} \frac{x'}{\gamma^3 x'^3} = \frac{q}{4\pi\varepsilon_0} \frac{1}{\gamma^2 x'^2}$

The E' field is "γ^2 suppressed" along the direction of travel

Along the y' axis, (i.e. $x'=0$) $\quad E'(0,y',0) = \frac{\gamma q}{4\pi\varepsilon_0} \frac{y'}{y'^3} = \frac{q}{4\pi\varepsilon_0} \frac{\gamma}{y'^2}$

The E' field is "γ enhanced" $\perp$ to direction of travel



- $\vec{E}$ points away from origin (at $t'=0$)
- Even at later t', it points away from wherever q is "now" in this frame
- Enhanced in sideway direction

This is the Lienard-Wiechart result of Ch. 10, done with much less pain!

$\vec{B} \neq 0$! It's easy to compute, + even to show that* $\quad \vec{B}' = -\frac{1}{c^2} \vec{V} \times \vec{E}'$

The $\vec{B}$ field arises purely from our Lorentz transformations

$$* \left[\begin{aligned} B_x' &= 0 \\ B_y' &= -\frac{v}{c^2} \gamma E_z = \frac{v}{c^2} E_z' \\ B_z' &= +\frac{v}{c^2} \gamma E_y = -\frac{v}{c^2} E_y' \end{aligned} \right]$$

$\vec{B}$ field is a "relativistic effect", and $\vec{E}$ field viewed from a different perspective!