3320 - ch 7.1

Our story so far has been static (although we do invoke current = moving charges... but even then, the current was steady).

Let's zoom in on current, as we move towards time dependence.

What makes current flow? In materials, there will surely be resistance to flow (scattering, losses to thermal motion)

So you need some force to maintain a current.

In some situation (e.g. "free electrons") you might imagine that a steady push would accelerate charges ⇒ increasing current, but in many (most!) materials, a steady push results in a steady flow

$$\boxed{\vec{J} = \sigma \vec{f}}$$

$\vec{f}$ → force per unit charge

$\vec{J}$ → current density

σ ↳ A material dependent constant, "conductivity"

It is not our old friend surface charge

(same symbol, totally new meaning here)

or, invert it: $\vec{f} = \rho \vec{J}$

ρ ↳ resistivity $= 1/\sigma$

(It is not volume charge density! Sorry, we're running out of symbols!)

3320 Ch 7.2

Now usually in E&M, $\vec{f} = \frac{\vec{F}}{q} = \vec{E} + \vec{v} \times \vec{B}$

(and usually, $\vec{v}$ +/or $\vec{B}$ are small enough to neglect)

So often

$$\boxed{\vec{J} = \sigma \vec{E}}$$

"Ohm's Law". It's not a law of nature (like Gauss' law), it's just a common, approximate relation found in many normal materials.

Comment: $\vec{F} = m\vec{a}$, so why doesn't a steady push yield an ever increasing flow? After all, $\vec{J} = \frac{\text{charge}}{\text{volume}} * \vec{v}$ (Notes, p. 13)

$$= \frac{\text{charge carriers}}{\text{volume}} * \frac{\text{charge}}{\text{carrier}} * v$$

$$= nq\vec{v}$$

It would if there were no damping. In real materials, e^-'s are like a gas, with large (random) $\vec{v}$'s. Applying a force cause a drift of that gas, but collisions tend to randomi(ze) (thermalize) the motion. Think of "terminal velocity" of a parachutist: steady force (mg) yield steady speed.

$V_{thermal}$ is big, V_{drift} is small. (Like a swarm of bees in a breeze)

So the steady force causes a steady $\vec{v}_{drift} \Rightarrow \vec{J} = nq\vec{v}_{drift}$

3320 ch 7.3

Comment: σ (or $\rho = 1/\sigma$) depends on the material.

Big σ = "good conductor". (Small f yield big flow!)

For Copper (used in most house wiring as a conductor)

$$\underbrace{\sigma_{cu} \approx 6 \cdot 10^7}_{} \quad \frac{\text{Coul}/\text{sec}\cdot m^2}{N/\text{Coul}} = \frac{C^2 s}{kg\, m^3} = \frac{1}{\text{Ohm}\cdot m} = \frac{1}{\Omega m}$$

Huge! Hence, "a conductor"

For wood, $\sigma_{wood} \simeq 10^{-(8 \text{ to } 11)}$, tiny, hence an "insulator"

For a resistor in a circuit, I suspect σ might be, oh, $10^3 /\Omega m$

i.e. somewhere "midrange".

Comment: Wait! I though $\vec{E} = 0$ inside metals!?

well... it is for static situations: $\vec{J} = \sigma \vec{E}$, so

if $\vec{J} = 0 \Rightarrow \vec{E} = 0$

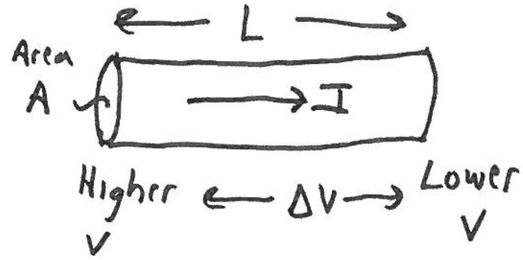
If $\sigma \to \infty$, then $\vec{E} = \vec{J}/\sigma \to 0$ even if (finite) current flo

For real conductors, σ is big (but finite). So yes, you do need small (but nonzero!) $\vec{E}$ to drive current in conductors

Comment: Power dissipated must be $= \underbrace{\Delta V}_{\frac{\text{work}}{\text{charge}}} \cdot \underbrace{I}_{\frac{\text{charge}}{\text{sec}}}$

3320 ch 7.4

Consider a uniform conducting wire



Use $\vec{J} = \sigma \vec{E}$

Here $J = \frac{I}{A}$ Here $E = \frac{\Delta V}{L}$ *, so $\Delta V = \underbrace{\frac{L}{A\sigma}} I$

[Call this $R = \frac{L}{A\sigma} = \rho \frac{L}{A}$

Resistance in Ω resistivity in $\Omega \cdot m$

Hey, $\Delta V = IR$, like in Phys 1120!

and Power $= \Delta V \cdot I = \frac{\Delta V^2}{R} = I^2 R$!

Real wires with current flowing do have a small E (and thus ΔV

The Big ΔV's will be across resistive elements in the circui

---

* Why was $E = \frac{\Delta V}{L}$ in that story? i.e, why is E uniform ?

If there was no material present

E would not be uniform!

The cylindrical conductor changes thing: Since $\vec{J}$ cannot leave the edges, $\vec{E} = \frac{\vec{J}}{\sigma}$ can't either. No "current leakage" changes the field!

This requires $\vec{E}_{inside}$ to be parallel to edges, $\vec{E} \cdot \hat{n} = 0$

Apparently a bit of charge must accumulate on surface to "shape" t

[Note: If $V = V_0 (1 - \frac{x}{L})$, this satisfies $\nabla^2 V = 0$ (check!) + $\frac{\partial V}{\partial n} = 0$

And $\vec{E} = -\nabla V = \frac{V_0}{L} \hat{x}$ (uniform) Uniqueness says this is the field

3320 Ch 7.5

In more complicated geometries, assuming current is <u>steady</u>
$\Rightarrow$ no charge buildup anywhere, then $\partial\rho/\partial t = 0$, so

$\vec{\nabla} \cdot \vec{J} = 0$ (Steady situations)

And Ohm's Law says, then, $\vec{\nabla} \cdot \vec{E} = 0$.

So in general, the problem with funny shaped resistors is

Find $\vec{E}$ such that $\vec{\nabla} \cdot \vec{E} = 0$ inside
$\vec{E} \cdot \hat{n} = 0$ at edge.

A "usual" boundary value problem like we've seen before.

Then, use $\vec{J} = \sigma \vec{E}$ and $I = \iint \vec{J} \cdot d\vec{a}$

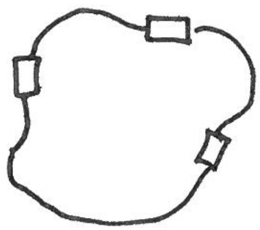
$\Rightarrow$ you will always find that $\Delta V = -\int \vec{E} \cdot d\vec{l} \propto I$

in other words, $\Delta V = IR$

[R may be hard to compute, it depends on the geometry, but once you have it, it's just a number!]

3320 Ch 7.6.

We've been looking at pieces of wire or resistors.
Let's step back and consider <u>complete</u> <u>circuits</u>.



→ Some circuit with various elements

e.g. conductors, resistors, batteries, PV panels, ... whatever!

If current flow around ... we've seen it must be <u>driven</u>

A <u>measure</u> of this is called EMF (Historically it's "Electromotive force" ... bad name, it's not a force!)

<u>Definition</u> $\mathcal{E} \equiv \mathrm{EMF} \equiv \oint \vec{f} \cdot d\vec{\ell}$ at one <u>instant</u> in ti...

a circuit loop

as usual, force / unit charge

This looks rather like <u>work</u> /unit charge, once around the loop?

$\vec{f}$ here can be <u>any</u> force that drives charges:

- <u>Electric</u> (so, $\vec{f} = \vec{E}$)
- <u>Magnetic</u> ($\vec{f} = \vec{v} \times \vec{B}$)
- Griffith's tiny ants with harnesses ($\vec{f} = \vec{F}_{ant}/q$)

Sometimes $\vec{f}$ will be mysterious to us (e.g. <u>inside</u> a battery, where chemistry & Quantum Mechanics make it hard to visualize the physics of $\vec{f}$)

· <u>Unit</u> of $[\mathrm{EMF}] = \left[\frac{\text{work}}{\text{charge}}\right] =$ Volts $= J/C$.

3320 ch. 7.7

If your circuit involved only electrostatic forces, then

$$\mathcal{E}_{\substack{\text{purely}\\\text{electrostatic}}} = \oint \vec{E}_{\text{electrostatic}} \cdot d\vec{l} = 0, \left(\text{since } \vec{\nabla}\times\vec{E}_{\text{e-static}} = 0\right.$$

So to drive current around a circuit, you need some other force!

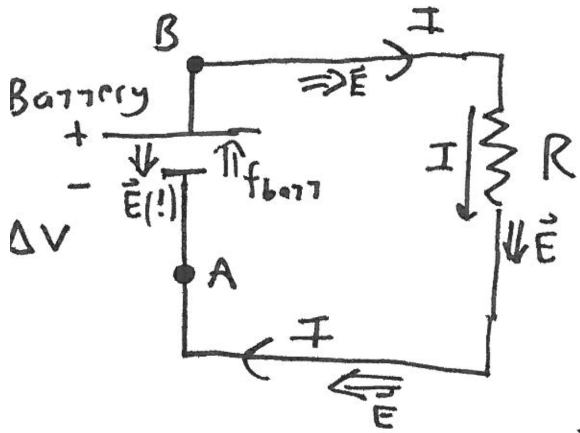
In battery, it's chemical.

In Van de Graff, it's mechanical

In Generator, it's magnetic ($\vec{v}\times\vec{B}$, "motional EMF")

↳ Back to this soon!

In this simple circuit:



$$\text{EMF} = \oint \vec{f}\cdot d\vec{l}$$

Here, the only "non electrostatic" $\vec{f}$ is inside the battery, f_{batt}.

There, charges are driven the wrong way from − side of battery to + side, by chemical forces we are not investigating.

- If flow (I) is steady, + there's no resistance inside battery, then $\vec{f}_{batt} = -\vec{E}$ (in order that $f_{net,\,in\ batt} = 0$, so charges don't accelerate, they're steady)

- Everywhere else in circuit, $\vec{f}$ is purely electrostatic, (just $\vec{E}$ at that point)

3320 ch 7.8

So $\mathcal{E}MF \equiv \oint \vec{f}_{total} \cdot d\vec{l} = \underbrace{\int_A^B}_{\text{through batt}} \left(\vec{f}_{batt} + \vec{E}\right) \cdot d\vec{l}$ ↱ chemistry!

$+ \underbrace{\int_B^A}_{\text{rest of circuit}} \vec{E} \cdot d\vec{l}$ ← no chemistry

so $\mathcal{E} = \int_A^B f_{batt} \cdot d\vec{l} + \underbrace{\oint \vec{E} \cdot d\vec{l}}_{\text{all the way}}$ — this is 0, no chemistry, just static $\vec{E}$ fields!

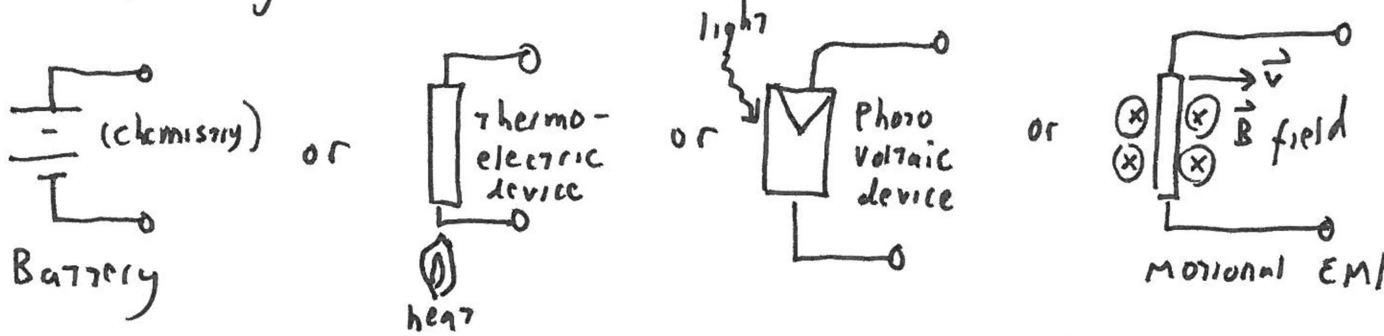
But since $\vec{E} = -\vec{f}_{batt}$ inside, the top eq'n (above) says

$$\mathcal{E} = \underbrace{\int_A^B 0}_{\text{through batt}} + \underbrace{\int_B^A \vec{E} \cdot d\vec{l}}_{\text{outside}} = -\underbrace{\int_A^B \vec{E} \cdot d\vec{l}}_{\text{outside}}.$$

But the last term is precisely $\Delta V_{battery}$!

[In Phys 1120 we speak of "EMF of a battery $= \Delta V$",
This is why! The EMF around the loop $= \Delta V_{outside}$]

Sources of EMF: anything that converts energy input into the ability to do work



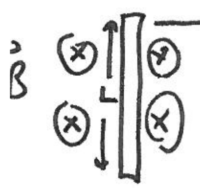
All these could do work, hook up 

Power $= I^2 R$
Energy $= I^2 R \cdot \Delta t$

3320 7.9

Let's consider Motional EMF : It's common (power generators!) + will lead us directly to Faraday's discoveries that changed the wo

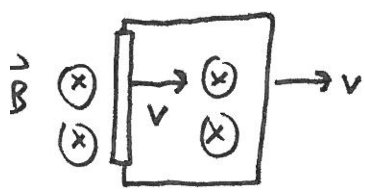
Consider 1st a metal bar moving in uniform $\vec{B}$



charges in bar feel $\vec{f} = \vec{v} \times \vec{B}$ (up for + charges)

This $\vec{f}$ pushes on charges, it creates an EMF!

Let's make a circuit to investigate this

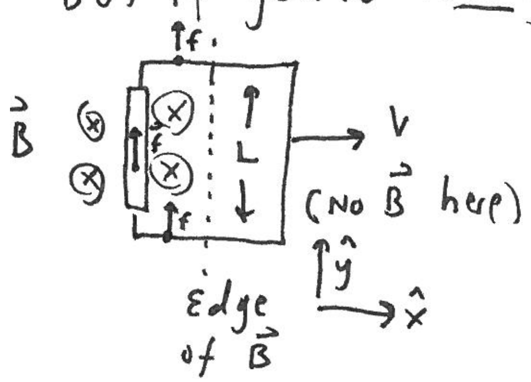


If $\vec{B}$ is uniform, nothing interesting happens.

Here, $\vec{f}$ is up in both left + right legs,

so $\oint \vec{f} \cdot d\vec{\ell} = 0$, the two edges cancel. No EM

But if you're leaving the field



Now $\text{EMF} = \oint \vec{f} \cdot d\vec{\ell} = VBL + 0 + 0 + ($

Let's go CW

$\vec{f} \perp d\vec{\ell}$ on top + bottom

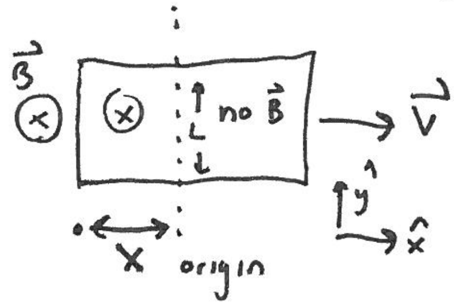
$\vec{f} = 0$ on righ

- If that right edge is "open" (circuit is "imaginary") then we have built up static + charge on top, − on bottom, and $\Delta V_{\text{open end}} = VBL$
- If that right edge is "closed" (e.g. a resistor R), then we will drive current around the loop, $I = \dfrac{VBL}{R_{Loop}}$

3320 7.10

<u>Note</u>: If loop shape, or $\vec{B}$ field, is complicated, $\oint \vec{f} \cdot d\vec{\ell}$ might be <u>hard</u> to calculate. But notice a cool thing in this example:

Let's examine "magnetic flux, Φ_B, through this circuit"



$$\Phi_B \equiv \iint \vec{B} \cdot d\vec{A} = B \cdot L \cdot |x|$$

and $$\frac{d\Phi_B}{dt} = BL\frac{d|x|}{dt} = -BLV$$

↳ − because if $V > 0$, $|x|$ is decreasing!

So in this <u>particular</u> case (loop <u>leaving</u> a uniform $\vec{B}$-field)

we have $\mathcal{E}\ (= BLV) = -d\Phi_B/dt$ the "Flux rule"

Turns out, $\mathcal{E} = -d\Phi_B/dt$ is <u>always</u> true. (See Griffith 296-7)

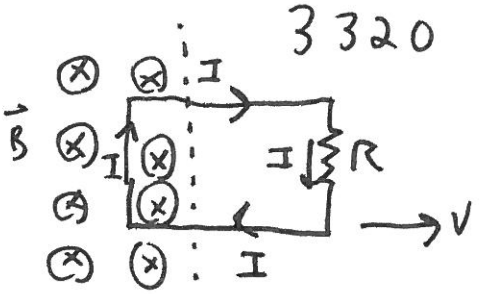
<u>Any</u> shape loop, any complicated $\vec{B}$ field, $\underline{\mathcal{E} = -d\Phi_B/dt}$

we have a new way to compute EMF, + it's often useful + easier!

<u>Comments</u> (1) For <u>solid</u> materials, moving through a non-uniform $\vec{B}$, these EMF's drive <u>currents</u>, called "Eddy Currents"

(2) <u>Sort</u> of looks like "drifting loops drive currents, without a battery" might violate energy conservation? Of course, it doesn't: Let's investigate!

3320 7.11.



Here $\mathcal{E} = BLV$, + $I = \frac{BLv}{R}$ flows.

But that means there is a magnetic force

$\vec{F}_{MAG} = I \int d\vec{\ell} \times \vec{B}$ on the left edge.

(upper + lower "left halves" have cancelling up & down forces, respectively)

thus $\vec{F}_{Mag} = ILB$ to the left. So, this "eddy current" here is creating drag. If you want steady I, you'll need an external

$\vec{F}_{ext} = ILB$ to the right (to maintain steady $\vec{V} \Rightarrow$ steady I)

$W_{by\ \vec{F}_{ext}} = \vec{F}_{ext} \cdot \vec{V} = +ILBV = \underline{BLV} * (\underline{I}) = BLV * \left(\frac{BLV}{R}\right)$

$W_{\text{dissipated in } R} = I^2 R = \left(\frac{BLV}{R}\right)^2 \cdot R = (BLV) * \left(\frac{BLV}{R}\right)$ equal!

Ahh! of course, power in (by fext) = power out (dissipated in load)

Energy is conserved, the EMF here drives current, but it's the Fext that supplies the energy!

In general, conducting loops "resist change in flux" by producing currents that interact with $\vec{B}$ field

e.g. Mag braking (trains, Prius, ...)

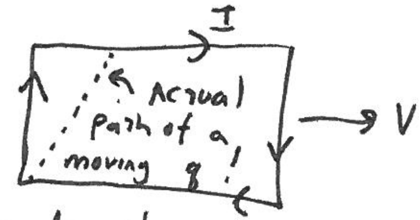
Magnet in metal tube (Coins in vending machines ...)
(= metal in mag tube!)

3320 7.12

Comment: Griffiths points out that $\mathcal{E} = \oint \vec{f} \cdot d\vec{l}$ at some time

But $\frac{\text{work done}}{\text{charge}} = \oint \vec{f} \cdot d\vec{l}$ following a charge around the circular path.

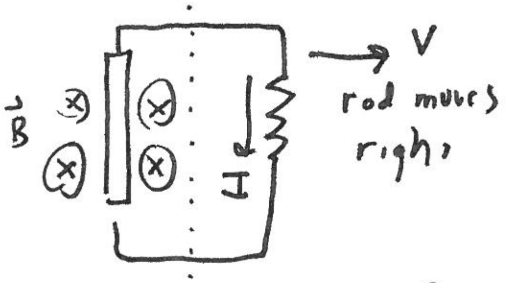
These can be subtly different, since e.g



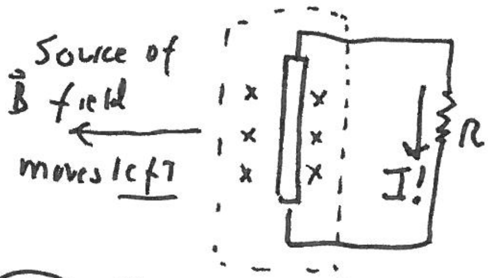
Griff (p.295-6) discusses this in some detail, it's really interesting if you are puzzling about how we can do work (e.g. heating a resistor) using a magnetic field, since $\vec{B}$ fields do no work on charges. We've just seen that in fact it is $\vec{F}_{ext}$ doing the work. $\vec{B}$ is essential here, without it the resistor won't heat up, but it is not the source of energy.

3320 7.13.

Let's consider several related (but distinct) situations: First,



① Motional EMF, the example we just did



② Here, $V_{rod} = 0$ (field moves

Situation 2 is different: $\vec{V} = 0 \Rightarrow q\vec{V}\times\vec{B} = 0$. No magnetic force on q's in rod... yet, relativity (just a frame shift, after all) says I must flow, there must be an $\mathcal{E}$ in situation 2 too.

Faraday experimented like this (in the 1830's!)

in ①, we say $\vec{f}$ is magnetic, arising from $\vec{V}\times\vec{B}$. The EMF is "MOTIONAL EMF"

in ② $\mathcal{E}$ must have the same value, but what is $\vec{f}$? Since $\vec{v} = 0$, it cannot be magnetic. It's subtle: turns out there is an $\vec{E}$ field in this reference frame! $\vec{E}$ and $\vec{B}$ are not "absolute", they depend on your frame. It's relativity at work, as we'll see later.)

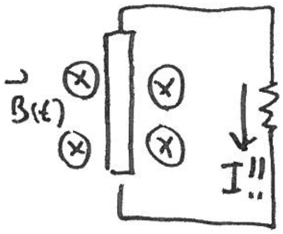
Exp'tally, $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ works for either setup.

(in ②, the moving magnet means flux through loop changes)

So the flux rule really becomes the tool of choice to find $\mathcal{E}$!

And then, Faraday considered an even more surprising case...

3320 7.14



In situation ③ here, $V_{rod} = 0$, and the magnet is totally stationary. Nothing moves, but e.g $B(t)$ just gets weaker (or stronger) with time.

The $\mathcal{E} = -\dfrac{d\Phi}{dt}$ still. This was Faraday's exp'tal outcome.

Nothing moves, not in any reference frame, so this is not motional EMF.

Apparently $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ is deeper than simply motional EMF

$\boxed{\text{Changing B fields drive currents}}$ It's a fact of nature. But How

Since the only force we know of that drives stationary charges is $\vec{E}$,

Faraday postulated a law of nature (see next page!)

$$\boxed{\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_{mag}}{dt}}$$

Faraday's law, integral form

This is the postulate: an induced $\vec{E}$ field is created, explaining the currents we observe in situation ③ (and 2) and many more!

Note: $\oint \vec{E} \cdot d\vec{l} \underset{\text{theorem}}{\overset{\text{Stokes}}{=}} \iint (\vec{\nabla} \times \vec{E}) \cdot d\vec{A}$

and $\Phi_{MAG} = \iint \vec{B} \cdot d\vec{A}$, so $\dfrac{d\Phi_{mag}}{dt} = \iint \dfrac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$

(* this last step works because, recall, $\mathcal{E}$ is defined for a fixed loop at some instant in time. So $d\vec{A}$ is not changing, only $\vec{B}$ might be! Thus, the $\dfrac{\partial}{\partial t}$ acts only on $\vec{B}$ in the integral.)

3320 7.15

What we have, then, is $\iint (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = -\iint \frac{\partial B}{\partial t} \cdot d\vec{A}$

This is true for any and all surfaces! That implies, then, at all point:

$$\boxed{\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t}$$

Faraday's Law in differential form
True at all points in space.

The ⊖ sign in Faraday's Law is called "Lenz' Law".
It reminds us that induced $\vec{E}$ points in the direction that (if it is allowed to drive currents) will "fight the change" in $\vec{B}$.

Side comment: I suppose Faraday could have postulated that case invokes a new force: changing $\vec{B}$ creates a "$\vec{G}$" field that also drives charges. But $\vec{G}$ would be just like $\vec{E}$, $\vec{F}_{new} = q\vec{G}$.

And $\vec{F}_{tot} = q(\vec{E} + \vec{G})$

$\vec{E}$: from static sources
$\vec{G}$: new, from changing B fields

This doesn't buy you anything, you can't distinguish $\vec{E} + \vec{G}$ in the lab...

And later, relativity will tell us $\vec{G}$ is not new/different... wait till the end of the term!

Still, our new "Faraday, induced" $\vec{E}$ is different, in that $\vec{\nabla} \times \vec{E} \neq 0$ any more. You can build "curly $\vec{E}$'s" in the lab (but not from static charges!)

3220 7.16

Faraday's Law is new... but the math is familiar!

Faraday : $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$

Ampere : $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

These are very similar! So, if you can solve Ampere's law problems, then (if $\rho = 0$)

and $\vec{\nabla} \cdot \vec{B} = 0$ (always)

$\vec{\nabla} \cdot \vec{E} = 0$ (when $\rho = 0$)

you can solve Faraday's law problems the same way!

In integral form

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \iint \mu_0 \vec{J} \cdot d\vec{A} = \mu_0 I$$

Again, if you can solve for $\vec{B}$ given $\vec{J}$ you can solve for $\vec{E}$ given $\Phi(t)$!

Changing magnetic flux "causes" (or is associated with) $\vec{E}$ fields in a very analogous way to how current "causes" (or is associate with) $\vec{B}$ fields.

Of course, this analogy only gives us the curly part of $\vec{E}$

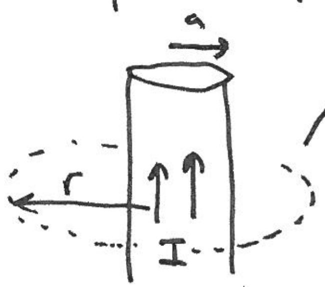
There may also be more, arising from $\rho \neq 0$!

3220 7.17

Let's do some examples!

Recall Ampere's law tricks when you have strong symmetry.

(Notes p.15) If current runs along a wire



Draw an Amperian loop.

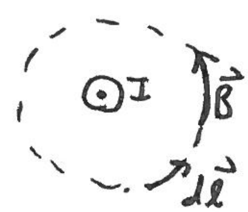
Argue, by symmetry, that B can only depend on r (distance from center), + must point in the azimuthal direction, (convince yourself, why

i.e. $\vec{B} = B(r)\,\hat{\varphi}$ ← purely by symmetry!

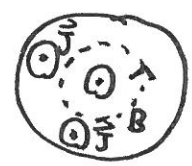
But then $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{thru}$

around the clever loop I picked, $\vec{B}\cdot d\vec{l} = \int B\,dl = B\cdot 2\pi r$!

So $B_{out} = \dfrac{\mu_0 I}{2\pi r}\,\hat{\varphi}$ if you are outside the wire.



If you are inside the wire,



+ $\vec{J}$ is uniform, then

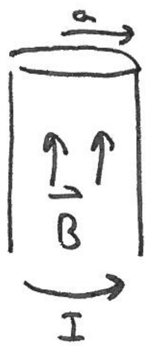
$I_{thru} = \iint \vec{J}\cdot d\vec{A} = J\cdot \pi r^2 = \dfrac{I}{\pi a^2}\cdot \pi r^2$, and you get

$B_{in} = \dfrac{\mu_0 I}{2\pi a^2}\, r\,\hat{\varphi}$ if you are in the wire.

Let's follow the same logic in a related Faraday problem!

3220 7.18

Consider a <u>solenoid</u> with n turns / unit length, current I.



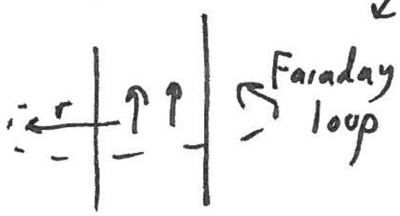
We <u>know</u> (notes p. 15) $\vec{B} = \mu_0 n I \hat{z}$ inside, 0 outside

This $\vec{B}$ field looks just like $\vec{J}$ inside the wire in the previous page!

If $I = I(t)$, then Faraday says $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_{\text{thru loop}}}{dt}$

So <u>same logic</u> as previous page says $\vec{E} = E(r)\,\hat{\varphi}$ by symmetry

and thus, for this loop $E \cdot 2\pi r = -\frac{d\Phi_{\text{inside } r}}{dt}$



So if you are <u>inside</u> the solenoid, just like on previous page,

$\Phi = \iint \vec{B} \cdot d\vec{A} = B \cdot \pi r^2 = (\mu_0 n I)\,\pi r^2$, and thus $\frac{d\Phi}{dt} = \mu_0 n \frac{dI}{dt} \pi r^2$

$\vec{E} = -\frac{1}{2\pi r} \cdot \mu_0 n \left(\frac{dI}{dt}\right)(\pi r^2)\,\hat{\varphi}$ <u>inside</u>.

(If I is increasing ↺, then $\vec{E}$ goes ↻ opposite... to fight the change!

If I is <u>decreasing</u> ↺, then $\vec{E}$ goes ↺ same way... to fight the <u>change</u>!

3220 7.19

If you are outside the solenoid, again, exact same logic:

$$\underbrace{\oint \vec{E}\cdot d\vec{l}}_{E\cdot 2\pi r} = -d\Phi/dt$$

$$E\cdot 2\pi r = -\frac{d}{dt}\left(\mu_0 n I \cdot \pi a^2\right)$$

→ B stops at $r=a$, so $\Phi = B_0 \pi a^2$, not $B_0 \pi r^2$!)

so $\vec{E} = -\mu_0 n \frac{dI}{dt}\frac{\pi a^2}{2\pi r}\hat{\varphi}$ (Same "direction" story as prev. page. Fight the change in flux!

This Result is stunning... $\vec{B}=0$ out there! Yet, magically, there is an $\vec{E}$ outside, generated by the changing $\vec{B}$ in the middle.

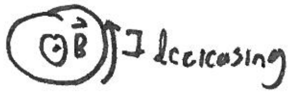
It's almost like you are communicating, without wires, by changing I over here, to create an $\vec{E}$ field that can move charges way over there... Hmmm, what could we do with this trick?!

See Griffiths (p.307) for another cool example, sort of similar, where you use a change $\vec{B}$ to mechanically drive a loop of uniform λ in a circle!

More comments: outside, $\vec{E} \propto \frac{1}{r}\hat{\varphi}$. This field visually "circles", but it has no curl out there! $\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} = 0$ here!

Just like $\vec{B} \propto \frac{1}{r}\hat{\varphi}$ outside wire, where $\vec{J}=0$, also "curves" but has zero curl! (The $\frac{1}{r}\hat{\varphi}$ field is special like this!)

3320 7.20



$\vec{E} \neq 0$ out here

Since $\vec{E} \neq 0$, sure looks like $V_A > V_B > V_C$ ... But this is crazy...

What if you go all the way 'round, + conclude $V_A > V_A$??

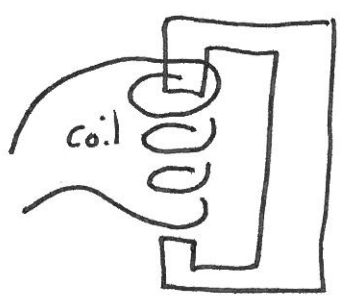
Well... V is not well defined if $\nabla \times E \neq 0$!

That's why we talk about EMF (not ΔV) around loops.

$\oint \vec{E} \cdot d\ell = 0$ if path contains no changing flux and thus $\int_A^B \vec{E} \cdot d\vec{\ell} = V_A - V_B$ is well defined

But if $\oint \vec{E} \cdot d\ell = -\frac{d\Phi}{dt} = \mathcal{E}$ path does contain changing flux then can get away with thinking about $\int_A^B \vec{E} \cdot d\vec{\ell} = V_A - V_B$ only if this path is clearly defined, it's not path independent, + voltage loses some of its conventional meaning!

Application:



Horseshoe magnet: If sound (air pressure) can mechanically wiggle the coil, thus changing flux, then you'll get $\mathcal{E} \Rightarrow$ drive current !

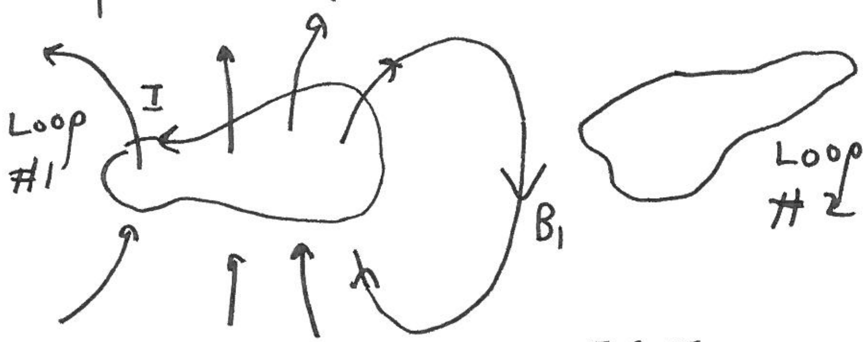
It's a microphone!

And, if instead you drive in a current $\Rightarrow \vec{F} = IL B$ will wiggle the coil, + you have a speaker! Same device! (yes, you can use your headphones as a microphone in a pinch, if they're built this w

3320 7-21

Application of Faraday's Law: INDUCTANCE

We will want to know how Faraday's Law "plays in" to circuits containing multiple wire loops, like:



- current flows in loop 1
- Creates B_1 everywhere
- Thus, flux ϕ_2 through loop 2
- If I_1 changes $\Rightarrow$ ϕ_2 changes $\Rightarrow$ induce an $\mathcal{E}$ in loop 2. (Once again, wireless communicatn)

In principle, Biot-Savart is the tool (certainly if I_1 is steady)

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\oint_{\text{Loop 1}} I_1 \frac{d\vec{l}_1 \times \hat{\mathcal{R}}}{\mathcal{R}^2}$$

[Here $\vec{\mathcal{R}}$ points from a source segment dl_1 in loop #1, to our point of interest, $\vec{r}$.]

could be nasty in practice... but must be doable in principle, say numerically on a computer. (Note, in particular, that $\vec{B}_1 \propto I_1$!)

Next: $$\underset{\text{loop 1}}{\Phi_{2\,(\text{due to})}} = \underset{\text{Loop 2}}{\iint} \underset{\substack{\uparrow \\ \text{from} \\ \text{loop 1}}}{\vec{B}_1} \cdot \underset{\substack{\uparrow \\ \text{area is} \\ \text{over loop 2}}}{d\vec{a}_2}$$ again, $\propto I_1$!

Let's define the proportionality constant: $\Phi_{2\,\text{due to }1} \equiv M_{21} I_1$

We call M_{21} the "Mutual inductance" (the reason for this name is more apparent on next page)

3320 7-22

So if I_1 changes $\Rightarrow$ Φ_2 changes $\Rightarrow$ $\mathcal{E}_2 = -\frac{d\Phi_2}{dt}$ will be present

Thus, changing I_1 drives (induces!) a current around loop 2!

$$\boxed{\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M_{21}\frac{dI_1}{dt}}$$

Note that M_{21} may be hard to compute, but if the loops are fixed in size, shape, + orientation, M_{21} is a constant

Specifically, $$M_{21} = \iint_{\text{Loop 2}} \frac{\vec{B}_1 \cdot d\vec{a}_2}{I_1} = \iint_{\text{Loop 2}} \frac{\mu_0}{4\pi} \oint_{\text{Loop 1}} \frac{d\vec{l}_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \cdot d\vec{a}_2$$

It's nasty looking, but purely geometry. Size, distance, orientation of loops

Could measure it in the lab, using formula in box above!

Griffiths uses Stoke's theorem (p. 311) to rewrite this formula, in the Neumann eq'n

$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint_{\text{loop 2}} \oint_{\text{loop 1}} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mathscr{r}}}$$

Still pure geometry. Still slightly intimidating to actually compute, but it's well defined (+ a computer could do it)

But it also has some magic in it: Swapping $1 \Leftrightarrow 2$ makes no difference in that last formula, so $M_{12} = M_{21}$

So we define "Mutual inductance" of two loops $= M = M_{12} = M_{21}$

What does this tell us, physically?

3320 7-23

Running I_1 in loop 1 induces $\Phi_2 = M_{21} I_1$

Running I_2 in loop 2 induces $\Phi_1 = M_{12} I_2$

If $M_{12} = M_{21}$, then 1 amp in loop one induces the same flux in #2

as running 1 amp in loop two induces in loop #1.

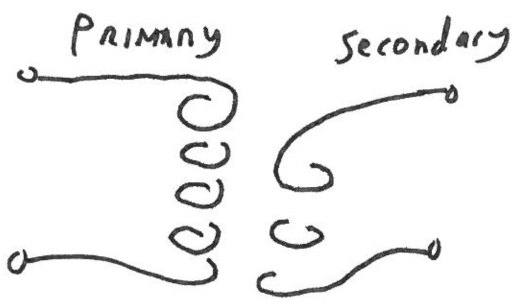
So if $\vec{B}$ from loop 1 is simple to find, find M_{21} (e.g, if loop 1 is a solenoid!)

If $\vec{B}$ from loop 2 " " " " " M_{12}.

Or, just measure it, once, either way.

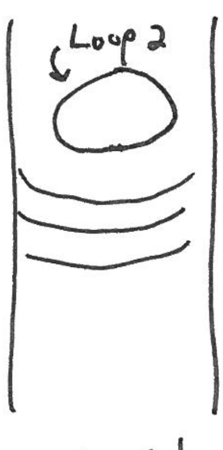
$$\mathcal{E}_{2,\text{ due to } 1} = -M \frac{dI_1}{dt} .$$

Transformers are built on this principle:



[Note, you need AC currents to get any $\frac{d}{dt}$, Transformers fail for DC!]

Example



Loop 1 = Solenoid

Toothbrush charges work this way!)

[Easy to find B of solenoid, thus ϕ_2 is easy, thus M_{21} is easy.

Hard to find B of single loop (+ thus ϕ_1)

But lucky for us, $M_{12} = M_{21}$, so we know what current in the little loop will induce in the solenoid after all!

3320 7-24

Even *more* magic happens when you realize changing I_1 *also* changes ϕ_1 itself. You can induce an EMF on *yourself*! Formally, it's all done: $\Phi_1 = M_{11} I_1$

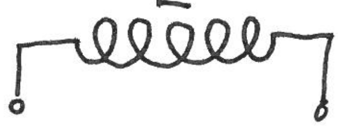
↳ "Self inductance", also called "Inductance", or L

So $\mathcal{E}_1 = -\frac{d\Phi_1}{dt} = -L\frac{dI_1}{dt}$ By definition, $L > 0$.

↳ Induced $\mathcal{E}$ "fights the change", it's "Back EMF"

Big L helps damp out high frequency noise, because high $f \Rightarrow$ large $\frac{d}{dt}$ $\Rightarrow$ large back EMF which fights the *change* + smooths things... (More on this later)

L

In diagrams,  Inductor

Units: $[L] = \frac{[\mathcal{E}]}{[dI/dt]} = \frac{V}{A/s} \equiv H$ for Henry. (Joseph Henry)

Also, since $\Phi_1 = L I_1$, and $[\Phi] = [B \cdot \text{Area}] = [T \cdot m^2]$
$\equiv$ Webers, W

Then $1 \text{ Henry} = \frac{1 \text{ Weber}}{\text{Amp}}$.

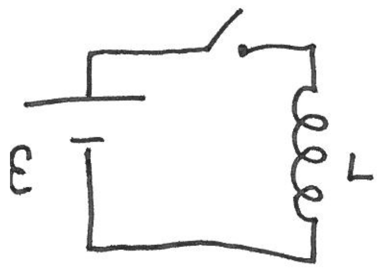
(e.g. look at Neumann eq'n to convince yourself)

Finally, note that $[\mu_0] = \frac{\text{Henry}}{m}$, $\underline{\underline{\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{Henry}}{m}}}$

(This is very handy for Electrical Engineers!)

3320 7-25

Energy in Magnetic Fields.



when you close the switch, I increases

(It doesn't jump because "back EMF" prevents rapid change.)

What is the work done to increase I by dI, working against this "back EMF"?

I know $\text{Power} = \frac{dW}{dt} = \Delta V \cdot I = \left(L \frac{dI}{dt}\right) \cdot I$

(If I is increasing, this is positive, we do + work against back ε)

So $W_{\text{to get } I \text{ from } 0 \text{ to } I_f} = \int \frac{dW}{dt}\, dt = \int_{I=0}^{I_f} L I \frac{dI}{dt}\, dt = \frac{1}{2} L I_f^2$

This is the answer, + it's different than the work done on a resistor, because this $\frac{1}{2} L I^2$ energy is stored, you can get it all back!

It reminds me of a similar story with Capacitors.

As you charge one up, $\frac{dW}{dt} = \Delta V \cdot I = \frac{Q}{C} \cdot \frac{dQ}{dt}$

So $W_{\text{to get from } 0 \text{ to charge } Q_f} = \int_0^{Q_f} \frac{Q}{C}\, dQ = \frac{1}{2} \frac{Q^2}{C} \quad \left(= \frac{1}{2} C (\Delta V)^2\right)$

Here, again, we store energy by charging up a capacitor

3320 7-26

n turns / length.

Example: In an ∞ solenoid, $B_{inside} = \mu_0 n I$.

so $\Phi = B \cdot \text{Area} \times \#\text{ turns}$ ↳ (Because each coil has $\phi = B \cdot A$!)

$= (\mu_0 n I) \cdot (\pi a^2)(\#\text{ turns})$

But we define $\Phi = L I$, so by inspection

$L_{\infty\text{ solenoid}} = \mu_0 n \cdot \pi a^2 \cdot (\#\text{ turns})$ (It's ∞, but $\frac{L}{\text{unit length}}$ is finite!)

To ramp current up from 0 to I (previous page) takes work

$$\text{Work} = \text{stored energy} = \frac{1}{2} L I^2 = \frac{1}{2}(\mu_0 n \pi a^2 \cdot \#\text{ turns}) \cdot I^2$$

But $I = \frac{B}{\mu_0 n}$, so

$$\text{stored energy} = \frac{1}{2}\frac{B^2}{\mu_0} \cdot \underbrace{\frac{\pi a^2}{n} \cdot \#\text{ turns}}$$

this is $\frac{\text{Area}}{\text{turns/length}} \times \text{turns} = \text{Area} \times \text{length} = \underline{\text{Volume}}$.

So Energy of ∞ solenoid is infinite, but

$$\frac{\text{Energy}}{\text{Unit volume}} = \frac{1}{2}\frac{B^2}{\mu_0}.$$

This turns out not to be an "accident", it's a general result!

See next page...

It's cool, it makes me think the solenoid stores energy, not because of the current, but because of ("in") the magnetic field !?

3320 7-27.

Let's enjoy a short orgy of vector-calculus to investigate that last claim. It's good practice (we'll do lots more of this kind of math) + the result is hugely important: B fields store (contain!) energy,

Loop:

$$\Phi_{loop} = \iint \vec{B}\cdot d\vec{a} = \iint \underbrace{(\vec{\nabla}\times\vec{A})}_{} \cdot d\vec{a}$$

→ Recall, the mag. vector potent

By Stokes! $\oint_{loop} \vec{A}\cdot d\vec{\ell}$

Now, by def., $\Phi = L I$, and from prev page, $W_{\text{to ramp I up}} = \frac{1}{2} L I^2$

So $W_{\text{to build up a current I in this loop}} = \frac{1}{2} I (LI) = \frac{1}{2} I \cdot \Phi = \frac{1}{2} I \oint \vec{A}\cdot d\vec{\ell}$

Let's write $\vec{A}\cdot d\vec{\ell}$ as $\vec{A}\cdot\vec{I}\, d\ell$, just a notation, so that I can generalize: Recall Griffiths 5.30: $\int \ldots \vec{I}\, d\ell = \iiint \ldots \vec{J}\, d\tau$ from line current to volume current

So this "rule" says $W_{\text{to build up } \vec{J} \text{ in space}} = \frac{1}{2} \underbrace{\iiint}_{\text{everywhere } \vec{J} \text{ flows!}} (\vec{A}\cdot\vec{J})\, d\tau$

This reminds me of an old electric result in chapter 2 (2.43)

$$W_{\text{to build up } \rho \text{ in space}} = \frac{1}{2} \iiint (\rho \cdot V)\, d\tau$$

↳ $\frac{1}{2}$ 'cause building from scratch

→ $\frac{\text{charge}}{\text{volume}} \times \frac{\text{energy}}{\text{charge}} = \text{Energy}$

We could stop here (+ say "currents in presence of $\vec{B}$ or $\vec{A}$ fields store energy) But, we can go further!

3320 7-28

Here's the nifty/trick part. Let's use Ampere's Law

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ to eliminate $\vec{J}$ from the story. This is a big idea!

Fields $(\vec{E}, \vec{B})$ are all you need to know (since Maxwell's Eq'ns tell you ρ + $\vec{J}$, given $\vec{E}$ & $\vec{B}$!)

so $W_{mag} = \frac{1}{2} \iiint \vec{A} \cdot \left(\frac{\vec{\nabla} \times \vec{B}}{\mu_0} \right) d\tau$ ← Got rid of $\vec{J}$, here

Front flyleaf identity #6 says $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

thus $\vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$

$= \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$

so $W_{mag} = \frac{1}{2\mu_0} \iiint B^2 \, d\tau - \underbrace{\frac{1}{2\mu_0} \iiint \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \, d\tau}$

By divergence theorem, this $= \frac{-1}{2\mu_0} \oint\!\!\oint_{\text{surface}} (\vec{A} \times \vec{B}) \cdot d\vec{a}$

We are now integrating over all space, so this surface is at ∞.

If $\vec{J}$ is localized, then $\vec{A}$ & $\vec{B}$ vanish at ∞ (faster than $\frac{1}{r}$!)

so this surface integral vanishes, leaving

$W_{mag} = \frac{1}{2\mu_0} \iiint_{\text{all space}} B^2 \, d\tau$. There's the same result again, B fields "store energy"!

3320 7-29

5. $u_{mag} \equiv \frac{\text{Energy}}{\text{Volume}}$ or "energy density" $= B^2/2\mu_0$

Comments: ① This reminds me of $u_{electric} = \varepsilon_0 E^2/2$!

② I used Ampere's law in this derivation. Ampere's law is not correct if you have a $\frac{\partial \vec{E}}{\partial t}$, but the formula above turns out to still correctly define magnetic energy density, in general.

③ It is curious that B (which "does no work") can store energy!!

But building up B ⇒ changing B ⇒ there is an induced E, and E does do work. (we started by find work done against the back EMF which arises from $\vec{E}$ fields)

④ we see (again & again) beautiful parallels between E + B, here $\frac{1}{2}\varepsilon_0 E^2$ and $\frac{1}{2\mu_0} B^2$ as "stored energy density" same mathematical structure. (It's no coincidence, of course!)