3320 RLC-1

Having studied INDUCTANCE, let's consider how we might deal with R, L, & C in real-life circuits.

Symbol	Circuit Relation	Geometry	Field
	$Q = CV$ or, $I = C\,dV/dt$	$C = \varepsilon_0 A/d$ (Parallel plates)	$E = \sigma/\varepsilon_0$ Different σ's!
	$V = IR$	$R = \rho L/A$ (uniform rod)	$J = \sigma E$
	$V = -L \frac{dI}{dt}$	$L = \mu_0 n A \cdot N$ turns (Long solenoid)	$B = \mu_0 n I$ or $\Phi_B = LI$

Kirchoff's Laws say $\sum_{\text{around any closed loop}} \Delta V = 0$ and $\sum I_{in} = 0$

all currents entering any point (node) in a circuit.

"Solving a circuit" means, generally, finding $I(t)$ through, +/or $\Delta V(t)$ across, all circuit elements

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Battery
$V = V_0$


AC voltage source
$V = V_0 \sin(\omega t + \delta)$


AC current source
$I = I_0 \sin(\omega t + \delta)$

Notes: $\varepsilon_0 = 8.85 \cdot 10^{-12} \frac{\text{farad}}{m}$ $\Rightarrow$ real life C's range from $\underbrace{<1\,pf}_{10^{-12} f}$ to $\underbrace{>1\,f}_{\text{needs serious dielectrics!}}$

- Typical Resistors range from $<1\,\Omega$ to several $M\Omega$

$\rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{H}{m}$ (and $L = \frac{\mu_0 N^2 A}{\ell}$) means real life L ranges typically from $<10^{-6}\,H \rightarrow \sim 1$ Henry.

(Although inductors $\Rightarrow \frac{dB}{dt} \neq 0 \Rightarrow$ might be worried that "voltage" doesn't mean anything, in fact it's generally OK. Outside the inductor, $B \simeq 0$, + usually it's well defined to talk about ΔV for paths outside the inductor - but use "EMF" when needed!)

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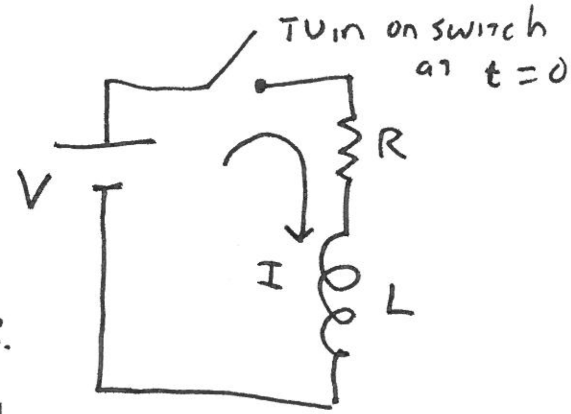

<u>Example</u>: A simple RL circuit

R might be distributed (in wires, battery, the inductor, etc). So might the inductance.

This is a model, + lets us see the general solution method.

(<u>Intuitively</u>, L doesn't "like" instant changes in current. So, we <u>expect</u> that at $t=0$, "I" <u>slowly</u> changes from 0... after a long time (steady state!) there are no more <u>changes</u> in anything, $\Rightarrow \Delta V_{inductor} = 0$, L acts like an ideal wire, + this is a simple (one R) circuit.)

Kirchoff says $$V - IR - L\frac{dI}{dt} = 0$$

I assume V, R, & L are known, + we want $I(t)$.

$$L\frac{dI}{dt} + IR = V$$ is a 1^{st} order, <u>in</u>homogeneous ODE.

There are several <u>methods</u> to solve this, I'll show two.

The first is <u>direct</u>, maybe <u>familiar</u> from Phys 2210. It gets unwieldy when there are lots of circuit elements.

The second is <u>tricky</u>, ("Phasors") + <u>very</u> powerful. We'll come back in a few pages + learn this method!

Sol'n Method #1

1) Find the <u>general</u> homogeneous sol'n to $L\frac{dI_H}{dt} + I_H R = 0$

2) Find <u>some</u> particular sol'n of the full (inhomog) eq'n

3) Add these, to get a full general sol'n

4) Figure out the (one) arbitrary constant in I_H using <u>initial</u> <u>conditions</u>.

This method works for $V = V_0$ (DC battery)

Also if $V = V_0 \cos \omega t$ (AC power supply)

and thus, by superposition, we can in fact solve for <u>any</u> periodic $V(t)$ (!!) because Fourier says (any periodic) $V(t) = \sum_n V_n \cos \omega_n t$

So our method will be pretty general!

Here, the homogeneous eq'n $\frac{dI_H}{dt} = -\frac{R}{L} I_H$

separates, $\frac{dI}{I} = -\frac{R}{L}dt \Rightarrow I_H(t) = I_H(t=0)\, e^{-Rt/L}$

an as-yet arbitrary, undetermined constant.

(As might be expected, over time, the resistor will <u>kill</u> <u>off</u> any starting current over time, but the inductor "stretches out" that time a bit.)

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For particular sol'ns, you don't need generality. "Guess & check" is a perfectly acceptable method!

So e.g. if $V = V_0$ = constant, $L\frac{dI_p}{dt} + I_p R = V_0$ = constant

you might <u>guess</u> perhaps $I_p(t) = a e^{-Rt/L} + b$ (? Maybe ?)

Let's see! If so, $\frac{dI_p}{dt} = -\frac{R}{L} a e^{-Rt/L}$, + our ODE says

$$L\left(-\frac{R}{L} a e^{-Rt/L}\right) + \left(a e^{-Rt/L} + b\right) R = V_0$$

Hey, these cancel

leaving $bR = V_0$. <u>Nice</u>. It <u>works</u>, as long as $b = V_0/R$

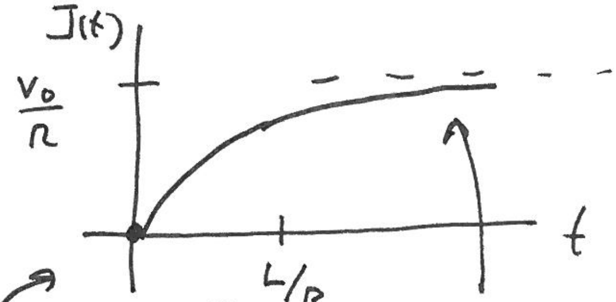
Recombinging with I_H now (+ calling $\underbrace{I_H(t=0) + a}_{\text{two unknowns}} \equiv c$)

$$I(t) = I_p + I_H = c e^{-Rt/L} + V_0/R.$$

c is our one undetermined constant: find it from initial condition, $I(0)=0$

So $c = -V_0/R$, and (for our <u>particular</u> problem)

$$I(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)$$



starts at 0, when close switch

("time constant")

<u>ends</u> as steady $I = V_0/R$, makes sense!!

It's more interesting with AC voltage supplies (+ that's what one usually has available!). So, let's solve this circuit for

$V = V_0 \cos(\omega t)$.

we *have* I_H already, just need a good guess for $I_p(t)$.

Well, sinusoidal driver might just result in sinusoidal current, so how about trying $I_p(t) = a \cos(\omega t + \varphi)$

Not sure what these are, let's see!

It's a *guess*, we try it! If it works, great!

so $L\, dI_p/dt + I_p R = V_0 \cos \omega t$ now.

plugging in our guess: $-La\omega \sin(\omega t + \varphi) + aR\cos(\omega t + \varphi) = V_0 \cos\omega t$

using standard trig identities, $\begin{cases} \sin(a+b) = \sin a \sin b + \cos a \cos b \\ \cos(a+b) = \cos a \cos b - \sin a \sin b \end{cases}$

typo! cosb! typo! sinb!

we get

$$\begin{aligned} &-La\omega \sin\omega t \cos\varphi \quad - aR \sin\omega t \sin\varphi \quad = \quad 0 + \\ &-La\omega \cos\omega t \sin\varphi \quad + aR \cos\omega t \cos\varphi \quad\quad\;\; V_0 \cos\omega t \end{aligned}$$

ooh! we can make this work, for all times, *if* we set coeff of $\sin\omega t$ on LHS $= 0$, + coeff of $\cos\omega t = V_0$,

i.e. *if* $-La\omega \cos\phi - aR\sin\varphi = 0$

and $-La\omega \sin\phi + aR\cos\varphi = V_0$

$\left.\right\}$ [Two eq'ns for two unknowns, a and φ!]

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The 1st eq'n says $-La\omega\cos\phi = aR\sin\varphi$

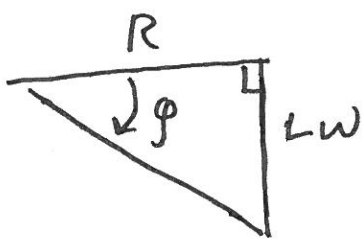
- would be true if $a=0$... but that makes $I_p = 0$, which is not useful to us.
- Or, it would be true if $\boxed{\tan\varphi = \frac{-L\omega}{R}}$. Aha! This is it,

If $\varphi = \tan^{-1}\left(-\frac{L\omega}{R}\right)$, then our I_p eq'n is satisfied.

Thus φ is not arbitrary, (+ is not "initial conditions"!) it's fixed by the circuit + driver constants.

The 2nd eq'n was $a(-L\omega\sin\phi + R\cos\phi) = V_0$.

The trick here is to draw a triangle showing $\tan\varphi = -\frac{L\omega}{R}$



From which I can just read off

$$\sin\varphi = \frac{-L\omega}{\sqrt{R^2+L^2\omega^2}} \quad \text{and} \quad \cos\varphi = \frac{+R}{\sqrt{R^2+L^2\omega^2}}$$

Putting that back in the 2nd eq'n will determine a for us:

$$a\cdot\left(\frac{+L^2\omega^2}{\sqrt{R^2+L^2\omega^2}} + \frac{R^2}{\sqrt{R^2+L^2\omega^2}}\right) = V_0, \quad \text{or} \quad \boxed{a = \frac{V_0}{\sqrt{L^2\omega^2+R^2}}}$$

We did it. We found $I_p = a\cos(\omega t + \varphi)$

Both of these are in boxes on this page

Thus, our full sol'n is

$$I(t) = I_P + I_H = a\cos(\omega t + \varphi) + I_{H0}\, e^{-Rt/L}$$

= persistent oscillatory response + Dying away piece.

Remember, a and φ are determined on prev page

I_{H0} is not, we can pick it still, to satisfy initial condition

So e.g if we closed the switch at $t=0$, so $I(0)=0$, then

$$I(t>0) = a\cos(\omega t + \varphi) - \underbrace{a\cos\varphi}_{\text{chosen to make } I(0)=0!}\, e^{-Rt/L}$$

with amplitude $a = \dfrac{V_0}{\sqrt{R^2+L^2\omega^2}}$ and <u>phase</u> shift $\varphi = \tan^{-1}\left(\frac{-L\omega}{R}\right)$

- If R is large, a is small. Makes sense, big R kills off long term currents.

- If $\omega = 0$ (battery!) $a \to V_0/R$ and $\varphi \to \tan^{-1} 0 = 0$

  we knew that already, for DC voltage, long times $\Rightarrow$ $I = V_0/R$, the inductor acts like an ideal wire when transients have died.

- If $\omega \to \infty$, $a \to 0$. Inductors don't like <u>rapid</u> changes of I, they suppress "response" at high <u>frequencies</u>!

(Summary: this method works fine! But with more R's, C's, or L's, series + parallel, this could rapidly get too painful....)

Method #2, "Phasors". This is a tricky method to solve the same problem. It turns out to be very powerful + very common. The trick converts all that nasty, painful sin & cos trig stuff to the simple algebra of exponentials! It uses

Euler's theorem: $e^{i\theta} = \cos\theta + i\sin\theta$, or

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

So exponentials really "contain" all the math of sin's + cos's, but so much simpler! E.g $\frac{d}{dt}\cos\omega t = -\omega\sin\omega t$ is a new, linearly independent function.

But, $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$, Just proportional. this saves a lot of grief.

So here's the trick. Instead of driving the circuit with $V = V_0\cos\omega t$, let's pretend we drive the circuit with $V_0 e^{i\omega t}$

A complex voltage? Well, in the end, we'll say

$$V_{TRUE} = Re\,(V_{fictitious}), \text{ and } I_{TRUE} = Re\,(I_{fictitious})$$

we can do this because the ODE is linear, so $Re\,(I)$ arises from $Re\,(V)$. Benefit: ODE becomes a (simple!) algebra problem

Cost: The sol'n looks complex, + we must remember to take the real part in the end!

Let's Redo the same problem again with this method.

So, $L\frac{dI}{dt} + IR = V(t) = \tilde{V}e^{i\omega t}$

we have this (fictitious) driving voltage, $\tilde{V}e^{i\omega t}$. It's complex.

- Real (physical) Voltage $= \mathrm{Re}(\tilde{V}e^{i\omega t})$
- If $\tilde{V}$ is itself a complex constant, i.e. if $\tilde{V} = V_0 e^{i\delta}$,

we can allow for drivers of the form $V_0(\cos \omega t + \delta)$, more general!

we once again really just need I_P. And again, we guess + check!

The guess here is quite simple, Try $I_P = \hat{I}e^{i\omega t}$ (See what happens!)

$L\frac{dI}{dt} + IR = \tilde{V}e^{i\omega t}$ is our ODE

Trial: $L\hat{I}(i\omega)e^{i\omega t} + \hat{I}Re^{i\omega t} = \tilde{V}e^{i\omega t}$.

Sweet! The $e^{i\omega t}$ cancels out, and we have an algebra eq'n

$L\hat{I}i\omega + \hat{I}R = \tilde{V}$, or $\underline{\hat{I} = \tilde{V}/R+i\omega L}$

That's really it. $\hat{I}$ is a constant, it's determined, +

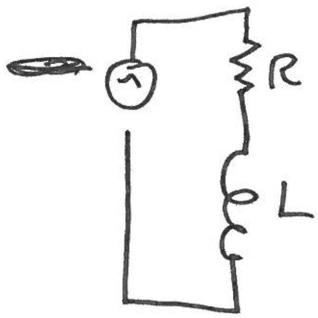
(then $I_{(True)} = \mathrm{Re}(I_{fictitious}) = \mathrm{Re}(\hat{I}e^{i\omega t}) + \underbrace{I_H}_{\text{(From Before)}}$)

We're done!

The sol'n looks like $\hat{V} = \hat{I}\,\hat{R}$, with $\hat{R}$ now <u>complex</u>.

we call it the imp<u>edence</u> or the com<u>plex</u> imp<u>edence</u>, $\underline{\underline{Z}}$.

So Here



we got $Z = R + i\omega L$

It's a simple <u>series</u> circuit, + the (complex) impedences just <u>add</u>.

This is a <u>general</u> rule! In <u>any</u> circuit, use

$Z_R = R$ impedence of resistor

$Z_L = i\omega L$ impedence of inductor

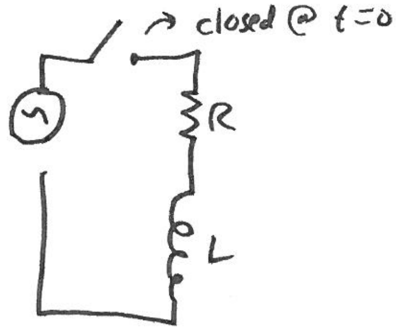
+ use your old "circuit simplification" tricks from Phys 2, like

Z's in series add, $1/Z$'s in parallel add up to $1/Z_{total}$ (etc)

(We'll come back to <u>capacitors</u> shortly, + prove

$Z_C = -\frac{i}{\omega C}$, + then you can add them in too!)

Let's return to our RL example and wrap it up:



we have $\hat{V} = \hat{I}(R + i\omega L)$

and $V_{true} = \mathrm{Re}\, \hat{V} e^{i\omega t}$

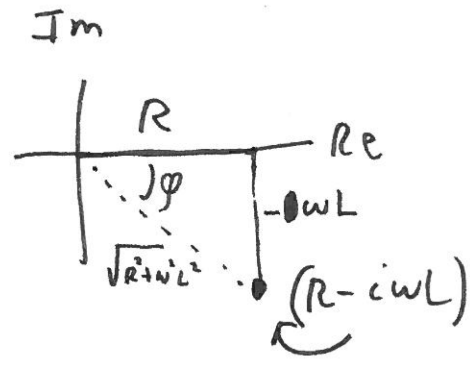
$I_{true} = \mathrm{Re}\, \hat{I} e^{i\omega t} = \mathrm{Re} \dfrac{\hat{V} e^{i\omega t}}{R + i\omega L}$

In our original setup, $V_{real} = V_0 \cos\omega t$, so $\hat{V} = V_0$.

So for that case: $I_{true} = \mathrm{Re}\left(\dfrac{V_0 e^{i\omega t}}{R + i\omega L}\right) \underset{\text{trick!}}{=} \mathrm{Re}\left(\dfrac{V_0 e^{i\omega t}}{R + i\omega L} \underbrace{\dfrac{R - i\omega L}{R - i\omega L}}_{\text{standard trick}}\right)$

so $I = \dfrac{V_0\, \mathrm{Re}}{(R^2 + \omega^2 L^2)}\left(e^{i\omega t}(R - i\omega L)\right)$

Next standard trick: Write $R - i\omega L$ in complex plane →



+ note that this point is simply

$\sqrt{R^2 + \omega^2 L^2}\, e^{i\phi}$ with, by inspection, ϕ as shown $= \tan^{-1} -\dfrac{\omega L}{R}$

So $I = \dfrac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \mathrm{Re}\left(e^{i\omega t} \cdot e^{i\phi}\right) = \dfrac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi)$

Exact same sol'n as we had before, (obtained just using $V = IZ$.)

Note: The method also works if $V = V_0 \cos(\omega t + \delta)$, just use $\hat{V} = V_0 e^{i\delta}$ (convince yourself!). Or, e.g. if $V = V_0 \sin\omega t$, just use $\hat{V} = V_0 e^{-i\pi/2}$ (or, for that matter, take $\mathrm{Im}(\,)$ instead of $\mathrm{Re}(\,)$)

What if the circuit has a capacitor?

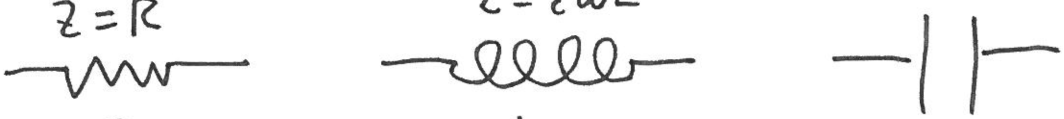
$V = \frac{Q}{C} \Rightarrow \frac{dV}{dt} = \frac{I}{C}$, or $I = C\, dV/dt$

so if $V = \hat{V} e^{i\omega t}$ then $I = C \cdot i\omega \cdot \hat{V} e^{i\omega t} = i\omega C\, V$

so this too looks like $V = I\text{"}R\text{"}$, but this time our complex impendence is $Z_c = \frac{1}{i\omega C}$ (Use "Z" instead of "R" whenever it's complex)

Summary

$Z = R$ $\qquad$ $Z = i\omega L$ $\qquad$ $Z = 1/i\omega C = \frac{-i}{\omega C}$

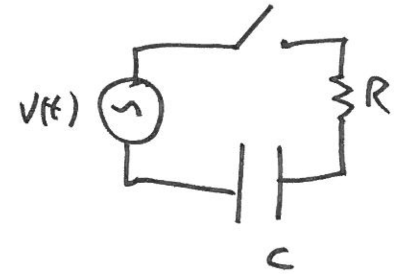


R $\qquad$ L $\qquad$ C

General procedure: Pretend you have a simple resistor circuit, using Z in place of R, and solve for V, I using usual phys 1120 (Kirchoff or "reducing equivalent resistors") methods.

That's it!

Example: An RC circuit



Turn it on @ $t=0$

Suppose $V = V_0 \cos \omega t$

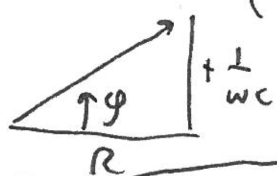
Think of this as two "Z"'s in series,

$$Z_{tot} = Z_R + Z_C = R - i/\omega C$$

then $\hat{V} = \hat{I} Z \Rightarrow \hat{I} = \dfrac{\hat{V}}{R - i/\omega C}$ and $I_{real} = V_0 \, \mathrm{Re} \, \dfrac{e^{i\omega t}}{R - i/\omega C}$

use usual trick, $I = V_0 \, \mathrm{Re} \left(\dfrac{e^{i\omega t}}{R - i/\omega C} \; \dfrac{R + i/\omega C}{R + i/\omega C} \right) = \dfrac{V_0}{R^2 + \frac{1}{\omega^2 C^2}} \, \mathrm{Re} \left(e^{i\omega t} \left(R + \frac{i}{\omega C} \right) \right)$

Draw a picture



$$\boxed{\varphi = \tan^{-1} 1/\omega C R}$$

tells me $e^{i\varphi} = \dfrac{R}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} + \dfrac{i/\omega C}{\sqrt{R^2 + 1/\omega^2 C^2}}$

so $I = \dfrac{V_0}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t + \varphi)$. That's it for $I_{particular}$

Still need I_{Homog}, from $0 = IR + Q/C$ or $\dfrac{dI}{dt} R = -\dfrac{I}{C}$

So $\dfrac{dI}{I} = -\dfrac{1}{RC} dt \Rightarrow I_H = I_0 e^{-t/RC}$

So $I_{tot} = \underbrace{\dfrac{V_0}{\sqrt{R^2 + 1/\omega^2 C^2}} \cos(\omega t + \varphi)}_{\text{steady sol'n.}} + \underbrace{I_0}_{\text{to be determined}} \underbrace{e^{-t/RC}}_{\text{transient, dies off}}$

Init condition: ΔV_{capac} cannot suddenly change! If it was 0 before, then after $t = 0^+$, we have $V_0 = IR$, or $I(t=0) = V_0/R$

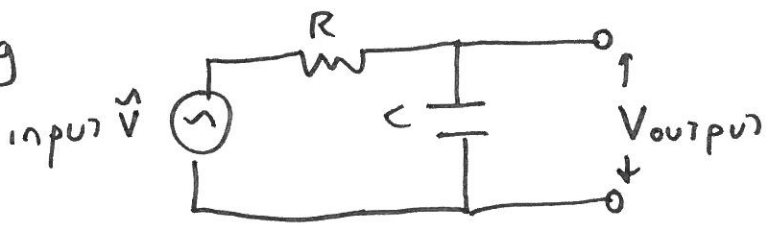
(I <u>can</u> jump instantly in this case!)

so $$I = \frac{V_0}{\sqrt{R^2 + 1/\omega^2C^2}} \cos(\omega t + \varphi) + \underbrace{\left(\frac{V_0}{R} - \frac{V_0 \cos\varphi}{\sqrt{R^2 + 1/\omega^2 c^2}}\right)}_{\text{Picked to make } I_0 = V_0/R} e^{-t/RC}$$

- Time constant is "RC"
- After several RC times, circuit oscillates @ driver frequency ω
- If $\omega \to 0$, $I_{\text{long term}} \to 0$, the capacitor <u>blocks</u> steady current!
- If $\omega \to \infty$, $I_{\text{long term}} \to \frac{V_0}{R}\cos(\omega t)$, like the capacitor isn't even there.

Many people prefer to have an output voltage as their signal, so

e.g



We <u>know</u> I in this circuit,

so $V_{out} = I Z_c = I(-i/\omega c)$

Thus $$V_{out} = \frac{V_{in}}{R - \frac{i}{\omega c}} \cdot \frac{-i}{\omega c} = V_{in} \cdot \frac{1}{1 + i\omega RC}.$$

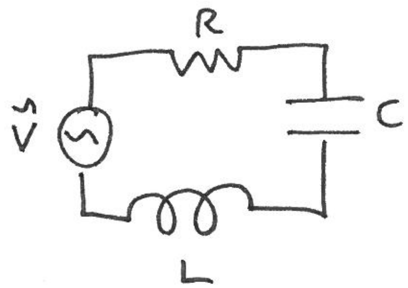
If $\omega \to 0$, $V_{out} = V_{in}$, the capac doesn't do anything

If $\omega \to \infty$, $V_{out} \to 0$. This is a "low pass filter",

it suppresses high frequency parts of the signal.

In general, <u>any</u> circuit like this is no harder than a phys 1120 circuit.

e.g



just use $Z = R - \frac{i}{\omega C} + i\omega L$

and then $\hat{V} = \hat{I} Z$

Parallel, series: just treat it like a simple resistor network.

(the power of phasors!)