3320 7-30

Summary to this point (we're getting close!)

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \quad \text{Gauss}$$

$$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t \quad \text{Faraday}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad \text{Ampere}$$

and $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

and $\vec{J} = \rho\vec{v}$ for moving charges.

Maxwell had all this in 1860's. He didn't invent these, he started here.

Could this be a complete theory of all E+M? We now have time depend.

+ we've used these to compute many exp'tally verifiable results!

Let's play a bit (mathematically) to investigate further, much like Maxwell did.

Note that for ANY vector field $\vec{G}$, it's always true $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{G}) = 0$

(Flyleaf #9, and fun + easy to prove yourself, try it!)

So e.g $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})$ must vanish, this is Mathematics, not physics.

But top of page says $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$, so we must have

$$\vec{\nabla}\cdot\left(\frac{-\partial\vec{B}}{\partial t}\right) = 0, \text{ and this means } \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$$

well, that's just fine, because $\nabla\cdot\vec{B} = 0$ at all times.

So indeed, looking good, no inconsistency between Math & Physics here..

Let's try this once more, but starting with $\vec{B}$...

3320 7-31

Math says $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) \equiv 0$. Does this agree with our physics?

Ampere says $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$, so we get $\vec{\nabla}\cdot(\mu_0\vec{J}) = 0$

Really? No, I don't think so! We've used many times

$\vec{\nabla}\cdot\vec{J} = -\partial\rho/dt$ (Griffith ch. 5.29)

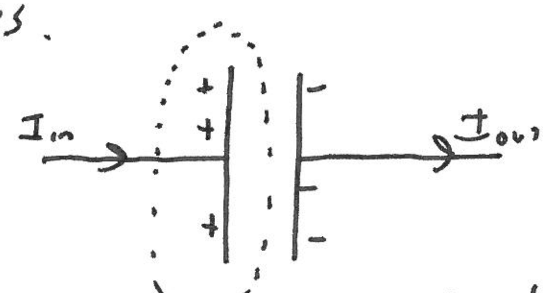
So, it's a problem! Maxwell realized that set of eq'ns cannot be the whole story, because they yield $\vec{\nabla}\cdot\vec{J} = 0$, always, which is physically not always the case!

If $\vec{\nabla}\cdot\vec{J} = 0$ always! then $\iiint_{\text{any Volume}} \vec{\nabla}\cdot\vec{J}\, d\tau = 0 \underset{\text{Theorem}}{\overset{\text{Divergence}}{\Longrightarrow}} \oiint_{\text{any surface}} \vec{J}\cdot d\vec{A} = 0$

$\vec{J}$ is current density, $\vec{J}\cdot d\vec{A}$ tells you "current leaving".

so $\oiint_{\text{all surfaces}} \vec{J}\cdot d\vec{A} = 0$ says "current in = current out", always, in all circumstances.

Really? Think about charging a capacitor plate up.




Q is building up (for a while), + thus in such (dynamic, time-dependent!) situations, $\oiint \vec{J}\cdot d\vec{A} \neq 0$, current goes in, none goes out, of dashed surface

Indeed, $\oiint \vec{J}\cdot d\vec{A}$ = net outflow of current $= -\dfrac{d}{dt}Q_{\text{enclosed}}$, physically, this is just conservation of charge

(same as $\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t$, right!)

3320 7-32

So conservation of charge says $\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t$

(Go back, even to p. 214 in Griffiths to review this!)

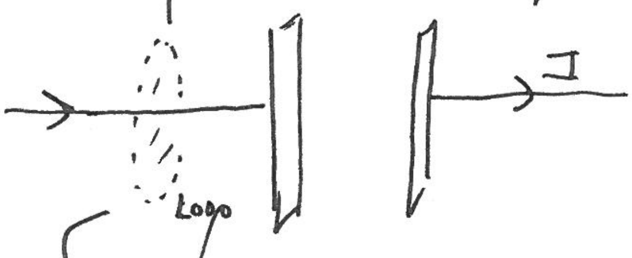
So Ampere's law forced us into $\mu_0 \vec{\nabla}\cdot\vec{J} = 0$, which is wrong in general.

(OK in statics, though!)

So Ampere's law is Maxwell's prime suspect.

Something wrong with it when $\partial\rho/\partial t \neq 0$ anywhere.

Let's push on that capacitor example:



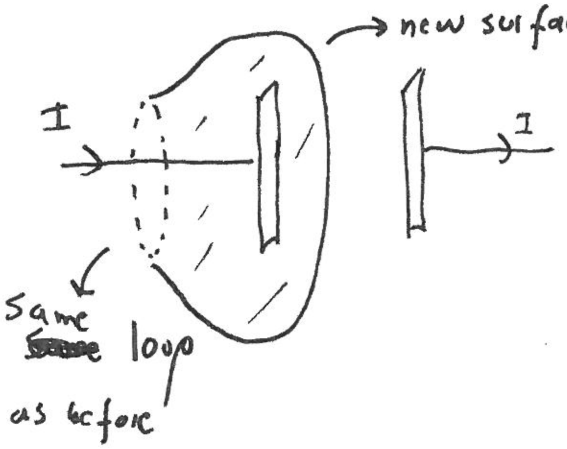

what's $\vec{B}$ here? Ampere says $\oint_{\text{Loop}} \vec{B}\cdot d\vec{l} = \mu_0 I_{thru} = \mu_0 \underbrace{\iint}_{\text{any surface bounded by that loop.}} \vec{J}\cdot d\vec{A}$

Seems like we'd get the usual $\vec{B}_{\text{long wire}}$, right?

But $I_{thru}$ can be computed using any surface bounded by that loop

Now, think about this "soap-bubble" surface, with same loop as bound

new surface, not the flat circle, shown shaded above.




But for this surface, $I_{thru} = 0$ !

Convince yourself, no current "pokes" the bubble surface!

3320 7-33

Ampere's law is failing:

$$\oint_{\text{loop}} \vec{B}\cdot d\vec{\ell} = \mu_0 \underbrace{\iint}_{\text{any surface bounded by loop}} \vec{J}\cdot d\vec{A}$$

Fails here, I came up with 2 different right sides, (one zero, one not) for same loop, same left side.

The problem arises because $\left.\frac{\partial \rho}{\partial t}\right|_{\text{inside bubble}} \neq 0$, charge builds up, so current in $\neq$ current out for the bubble-loop

(If currents were steady $\Rightarrow \frac{\partial\rho}{\partial t} = 0$ everywhere, we'd have no issue)

Let's follow Maxwell's nose...

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ Must be true, that's Math!

If $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$, we get $\vec{\nabla}\cdot(\mu_0\vec{J}) = 0$ instead of what current conservation says, namely $\vec{\nabla}\cdot(\mu_0\vec{J}) = -\mu_0\,\partial\rho/\partial t$

Can we fix Ampere's law? It's easy enough! Add $\vec{X}$ to the right s

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \vec{X} \qquad \text{Just try it!}$$

$$\text{so}\quad \vec{\nabla}\cdot(\vec{\nabla}\times B) = \mu_0\,\vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{X}$$

$$\underbrace{0}_{\text{Math}} = \underbrace{-\mu_0\,\partial\rho/\partial t}_{\text{current conservation}} + \vec{\nabla}\cdot\vec{X}$$

I can't yet spot what $\vec{X}$ should be... but if I use Gauss' law, which after all generated no troubles 3 pages ago, so should be OK, we get some more magic...

$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$ is gauss' law, so $\frac{\partial\rho}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t} \vec{\nabla}\cdot\vec{E}$
$= \varepsilon_0 \vec{\nabla}\cdot \partial\vec{E}/\partial t.$

So, we've got (putting it together with bottom of last page)

$$0 = -\mu_0\varepsilon_0 \vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}\cdot\vec{\boxtimes}.$$

Ah! Now I see it... if $\vec{\boxtimes} \equiv +\mu_0\varepsilon_0 \partial\vec{E}/\partial t$, this eq'n will always be satisfied.

This was Maxwell's "guess", a new + improved Ampere's law

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0 \partial\vec{E}/\partial t}$$

The extra term is $\mu_0\vec{J}_D$ with $\vec{J}_D \equiv \varepsilon_0 \partial\vec{E}/\partial t$ = "Displacement current"

It has units of current density, but it's not a physical flow of charge.

It's not a current! The name was Maxwell's (for historical reasons, ask me in office hours), + now is just a vestige.

Note: In statics, $\partial\vec{E}/\partial t = 0$ + we're back to the old Ampere's law

Why didn't Ampere notice this? Well, $\varepsilon_0\mu_0 = \frac{1}{9\cdot 10^{16}} s^2/m^2$ is a tiny #!

This new correction term was exp'tally undetectable in those days.

Only if $\partial\vec{E}/\partial t$ is huge does it matter... It just turns out that $\partial\vec{E}/\partial t$ can be huge in very common situations, as we'll see, but Ampere didn't think of the right exp't to ever "see" this!

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Look at the lovely symmetry now:

Just as "changing $\vec{B}$" creates a curly $\vec{E}$ (Faraday)

now "changing $\vec{E}$" creates a curly $\vec{B}$ (Maxwell's addition to Ampere)

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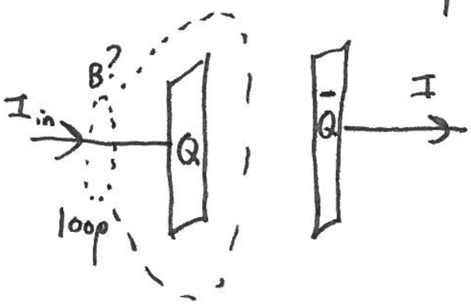
Look back at our "problem case" with the charging capacitor




Inside the capacitor, $E \approx \dfrac{Q}{\text{Area} \cdot \varepsilon_0}$ (Gauss)

so $\mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \dfrac{\mu_0}{\text{Area}} \cdot \dfrac{dQ}{dt} = \dfrac{\mu_0 I_{in}}{\text{Area}}$

So, if we use $\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_{thru} + \mu_0 \varepsilon_0 \iint_{\text{Area}} \dfrac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$

If your area on RHS is the flat area, $\vec{E} \approx 0$ there (outside capacitor!)

and r.h.s. is $\mu_0 I$

If instead you choose the bubble area, now $I_{\substack{poking \\ thru}} = 0$, but

2nd term is $\iint \dfrac{\mu_0 I}{\text{Area}} \cdot dA = \mu_0 I$, same result now!

Sweet! the problem (Different results depending on choice of area)

is resolved by adding in the "displacement" term.

Going backwards, if you start from our new + improved Maxwell Eq'ns then from $0 = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})$ (pure math), you now discover

$$\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t$$

In other words, Maxwell's Equations imply/require/give you charge conservation. We don't tack that on as an extra fact of nature, it's BUILT IN TO Maxwell's Eq'ns.

(So, as we'll see, is relativistic invariance also "built in".

This set of eq'ns IS NOW complete! It's a full, self consistent classical field theory. And, we'll see (soon) that it leads to still more, like energy & momentum conservation too. It's remarkable, the greatest synthesis in physics, possibly ever, certainly in 1865!

$$\boxed{\begin{aligned}
\nabla\cdot\vec{E} &= \rho/\varepsilon_0 \\
\nabla\cdot\vec{B} &= 0 \\
\vec{\nabla}\times\vec{E} &= -\partial\vec{B}/\partial t \\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{J} + \mu_0\varepsilon_0\,\partial\vec{E}/\partial t
\end{aligned}}$$

3320 7-37

One last thing ... (to be developed soon!)

We learned a lot by taking $\vec{\nabla}\cdot$ (Maxwell's curl eq'ns)

what about trying $\vec{\nabla}\times$? Let's try it ...

Use Front flyleaf again, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E}$

Now consider totally empty space, with $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = 0$. so

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\vec{E}.$$

But since $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$, we also have L.H.S $= -\vec{\nabla}\times\left(\frac{\partial\vec{B}}{\partial t}\right)$

$$= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

AND, By our (new) Ampere/Maxwell law,

the LHS $= -\frac{\partial}{\partial t}\left(\mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$. Again, in empty space, $\vec{J}=0$, so

Empty space: $-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = -\vec{\nabla}\vec{E}$. ← [Same result for $\vec{B}$, if you start with $\vec{\nabla}\times(\vec{\nabla}\times\vec{B})$]

For each component of $\vec{E}$, then e.g $\nabla^2 E_z = \mu_0\varepsilon_0\,\partial^2 E_z/\partial t^2$

Seen that before? In 1-Dim, $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ is the wave eq'n,

solved by traveling waves of speed v,

So in empty space, we will find sol'ns for $\vec{E}$ (+$\vec{B}$) that are non zero,

but are traveling waves with speed $v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\cdot 10^8$ m/s

Yikes! we have got to follow up on this, don't you think?!

Ch 7 wrapup -1-

Griffiths wraps up Ch.7 with Maxwell's Eq'ns in matter. There is nothing fundamentally new here (our set of Maxwell eq'ns in vaccuum is exactly correct still), it's just more convenient to shift notation with real materials, because $\rho$ and $J$ have two pieces in matter, e.g. $\rho = \underbrace{\rho_{free}}_{\text{charges you might measure or control or choose}} + \underbrace{\rho_{bound}}_{\text{charges that simply arise from polarization of material}}$

In Ch.4 you learned $\rho_B(\text{static}) = -\nabla \underline{P} \rightarrow$ Polarization

this is still true even if $\underline{P} = \underline{P}(t)$, $\rho_b(t) = -\vec{\nabla} \underline{P}(t)$

So, $\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \underline{P}$

and $\underbrace{\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0}_{\text{Always!}} = \dfrac{\rho_f}{\varepsilon_0} - \dfrac{\vec{\nabla} \cdot \vec{P}}{\varepsilon_0}$. So as in ch.4

Define $\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}$, + you immediately get $\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$

- So Gauss' law for $\vec{D}$ fields is same as before, even if $\rho_f$ is time dependent.
- $\vec{\nabla} \cdot \vec{B} = 0$ is also still true
- $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ is also still true (doesn't involve any flow of currents, unchanged in materials)

Ch 7 wrapup -2-

There's one last Maxwell Eq'n, involving currents (Ampere - Maxwell)

For currents, there is a <u>new</u> <u>twist</u> when things are $t$-dependent

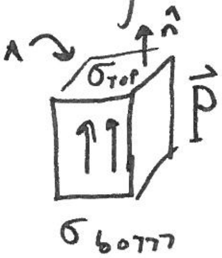
$\vec{J}$ has <u>three</u> parts: $\vec{J}_{free}$, $\vec{J}_{Bound} = \vec{\nabla} \times \vec{M}$, + something new!

$\vec{J}_{free}$ ↓ actual motion of free charges

$\vec{J}_{Bound}$ ↓ effective current due to spatial variation in Magnetization, just like in Ch. 6.

The "something new" is this: If polarization $\vec{P}$ depends on time, then surface charges $\sigma_b = \vec{P} \cdot \hat{n}$ also vary in time, so there <u>must</u> be a flow of bound charge!

It's a "Polarization current", it has nothing to do with magnetization! we call it $\vec{J}_P$




Recall $\sigma_{bound} = \vec{P} \cdot \hat{n}$ and $\rho_{bound} = -\vec{\nabla} \cdot \vec{P}$

If $\vec{P}$ varies, $dI = \dfrac{\text{charge flow}}{\text{time}} = \dfrac{\sigma_{Top}(t+\Delta t) - \sigma_{Top}(t)}{\Delta t} \cdot A$

$$dI = \frac{\vec{P}(t+\Delta t)\cdot\hat{n} - \vec{P}(t)\cdot\hat{n}}{\Delta t} A = \frac{\partial \vec{P}}{\partial t} \cdot \hat{n}\, dA$$

so $$\vec{J}_P = \frac{dI}{da_\perp} = \frac{\partial \vec{P}}{\partial t}$$

This is a real current, it's the physical (albeit bound) charges that are moving! So it too should contribute to $\vec{B}$ fields...

Ch 7 wrapup -3-

So, consider $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ } Always true, even in matter!

In matter, $\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_f + \vec{J}_B + \vec{J}_P) + \mu_0 \varepsilon_0 \partial \vec{E}/\partial t$

$$\vec{\nabla} \times \vec{B} = \mu_0 (\vec{J}_f + \vec{\nabla} \times \vec{M} + \partial \vec{P}/\partial t) + \mu_0 \varepsilon_0 \partial \vec{E}/\partial t$$

so $\vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial}{\partial t} \underbrace{(\varepsilon_0 \vec{E} + \vec{P})}_{\text{we define this to be } \vec{D}}$

this is $\mu_0 \vec{\nabla} \times \underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}$

As before, we define this to be $\vec{H}$. Cancel common $\mu_0$ to get

$$\underline{\vec{\nabla} \times \vec{H} = \vec{J}_f + \partial \vec{D}/\partial t}$$

This is Maxwell-Ampere in matter. Everything is time dependent.

↳ sometimes called $\vec{J}_D$ or "displacement current"

For linear materials, $\vec{D} = \varepsilon \vec{E}$

$$\vec{H} = \frac{1}{\mu} \vec{B}$$

- Maxwell's Eq'ns in vacuum (without $\vec{D}$ or $\vec{H}$) are still correct, general, always true (classically)
- The new Maxwell's eq'ns in materials (this page + last) are <u>also</u> correct, general, + more help<u>ful</u> if material polarizes or magnetizes

In either form, these eq'ns are relativistically correct, + complete (no corrections, besides quantum effects, have been found in >150 yrs)

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Just as we derived boundary conditions by considering small boxes or squares around surface charges or currents in vacuum, so too you can easily derive Boundary Conditions in matter. Just start from our modified eq'ns and follow your nose.

E.g. $\vec{\nabla}\cdot\vec{D} = \rho_f$, so a small cube gives $D^{\perp}_{above} - D^{\perp}_{below} = \underbrace{\sigma_f}_{}$ only free surface charges matter,

Still get $E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

Still get $B^{\perp}_{above} - B^{\perp}_{below} = 0$.

One new one; from Maxwell-Ampere. The $\partial\vec{D}/\partial t$ term doesn't do anything in the limit of "tiny squares" (flux of $\vec{D}$ will vanish in a small square, $\vec{D}$ isn't blowing up!), so

$$H^{\parallel}_{\cancel{below}\ above} - H^{\parallel}_{below} = \vec{K}_f \times \hat{n}$$

↳ Just free surface currents matter.

we'll come back to these soon enough, when light (EM waves!) strikes a real surface (like a mirror)