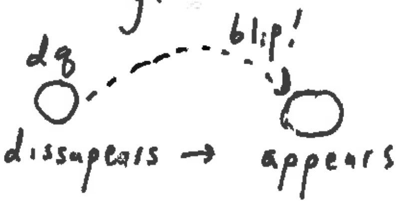
Conservation Laws for EM fields

we know electric charge is conserved:

GLOBALLY: total Q in universe doesn't ever change.

If this was all we knew, we might imagine:



This would conserve charge... but it's not how the world is!

Locally:

If charge leaves a volume, it must flow past the boundary! This is expressed mathematically by:

At a point: $$\left.\underbrace{\partial\rho/\partial t}_{\text{increase in charge/vol}} = -\underbrace{\vec\nabla\cdot\vec J}_{\text{(outflow of current density)}}\right\} \text{Local charge conservation}$$

Recall, $\vec J = \rho\vec v$, so $\vec J$ is a vector flow $\frac{\text{of charge}}{(\text{area})_\perp \cdot \text{time}}$

For a volume:

$$\underbrace{\frac{dQ}{dt} = \frac{d}{dt}\iiint \rho\, d\tau}_{} = -\iiint_V \vec\nabla\cdot\vec J\, d\tau \underset{\text{theorem}}{\overset{\text{DIVERG.}}{=}} -\oint\!\!\oint \vec J\cdot d\vec A = -I_{out}$$

$\frac{\text{increase of charge}}{\text{time}}$ = - outflow of current

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Is anything else conserved locally? I would expect:

Energy, momentum, + angular momentum

↑ we'll focus on this

↑ touch on this

↑ mention this!

In general, "conservation of X" means $\frac{\partial X}{\partial t} = -\vec{\nabla}\cdot\left(\text{volume flow of a current associated w. } X\right)$

Let's summarize some relevant ideas from last chapter:

① Stored Electrical Energy $W_e = \frac{1}{2}\varepsilon_0 \iiint E^2\, d\tau$ = work (energy) required to assemble charges to build this E field

or

Electric Energy Density $w_e = \frac{1}{2}\varepsilon_0 E^2$ = $\frac{\text{energy}}{\text{unit volume}}$ "stored" in E-field at a point

② Stored Magnetic Energy $W_B = \frac{1}{2\mu_0}\iiint B^2\, d\tau$ = work (energy) required to get currents flowing (against back EMF's!) that build this B-field

or

Magnetic Energy density $w_B = \frac{1}{2\mu_0} B^2$ = $\frac{\text{energy}}{\text{unit volume}}$ "Stored" in B-field at a point

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Then,

$$U_{\text{Total, EM}} = \iiint \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) d\tau$$ = Total stored EM energy in fields

or

$$u_{\text{Total, EM}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$ = Stored local EM energy/unit volume

≡ "Energy density"

Conservation of Energy ⟺ we're looking for a relation that looks like

$$\frac{\partial}{\partial t}(\text{energy density}) = -\frac{\underline{\text{outflow}}}{\text{volume}}\ \text{of some energy }\underline{\text{current}}$$

$$= -\vec{\nabla}\cdot\left(\text{"energy current density"}\right)$$

what would this be? ↗ Let's figure it out!

Consider a situation with charged particles + currents that produce $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. Let's zoom in on one of the charges "dq" moving around with velocity $\vec{V}$ at time t.

$$\underbrace{dW_q}_{\text{EM work done ⟺ on } q \text{ by fields}} = \vec{F}_{on\,q}\cdot \underset{\substack{\uparrow\\ \text{distance}\\ \text{traveled}}}{d\vec{\ell}} = \underbrace{dq\,(\vec{E}+\vec{v}\times\vec{B})}_{\substack{\uparrow\\ \text{This is } \vec{F}_{on\,q}}}\cdot\underbrace{(\vec{v}\,dt)}_{\substack{\uparrow\\ \text{this is } d\vec{\ell}\,!}}$$

$$= dq\,\vec{E}\cdot\vec{v}\,dt \qquad \left(\text{Since } \vec{B} \text{ does no work!}\right)$$

$$\text{so}\quad \frac{dW_q}{dt} = dq\,\vec{E}\cdot\vec{v} = \underbrace{\rho\,d\tau}_{\text{this is } dq}\ \vec{E}\cdot\left(\frac{\vec{J}}{\rho}\right) \rightarrow \left(\text{this is } \vec{v}\text{, since } \vec{J}=\rho\vec{v}\right)$$

For $\underline{\underline{\text{many}}}$ charges, adding this up gives

Globally: $dW_q/dt = \iiint(\underbrace{\vec{E}\cdot\vec{J}})\,d\tau \rightarrow$ This is the "EM power density", Joules/sec·m^3

Locally: $\partial u_q/\partial t = \underbrace{\vec{E}\cdot\vec{J}} \rightarrow \dfrac{\text{EM work done on charged particles}}{\text{volume}}$

A common trick in this course is to eliminate charges (ρ) + currents ($\vec{J}$) using Maxwell's eq'ns, in favor of pure fields.

Here, e.g, let's use $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \partial \vec{E}/\partial t \leftrightarrow$ Solve for $\vec{J}$!

$$\text{So } \vec{E} \cdot \vec{J} = \frac{\vec{E}}{\mu_0} \cdot \left(\vec{\nabla} \times \vec{B} - \mu_0 \varepsilon_0 \partial \vec{E}/\partial t \right)$$

$$= \frac{\vec{E} \cdot (\vec{\nabla} \times \vec{B})}{\mu_0} - \varepsilon_0 \vec{E} \cdot \partial \vec{E}/\partial t \qquad \leftarrow (a)$$

Here's an (inobvious !?) step, using from flyleaf product rule #6

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

← Math, true for any fields $\vec{E}$ & $\vec{B}$

So the 1st term in (a) above is $\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot (E \times \vec{B})$

Now use Faraday's law, $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$ to get, from (a),

$$\vec{E} \cdot \vec{J} = -\frac{\vec{B} \cdot \partial \vec{B}/\partial t}{\mu_0} - \varepsilon_0 \vec{E} \cdot \partial \vec{E}/\partial t - \frac{\vec{\nabla} \cdot (\vec{E} \times \vec{B})}{\mu_0}$$

Here's a second (inobvious !?) step : in general, $\frac{\partial}{\partial t} \vec{A}^2 = 2 \vec{A} \cdot \frac{\partial \vec{A}}{\partial t}$

This trick is used twice above, once w. $\vec{E} \cdot \partial \vec{E}/\partial t$, once w. $\vec{B} \cdot (\partial \vec{B}/\partial t)$

$$\vec{E} \cdot \vec{J} = -\frac{\frac{1}{2} \frac{\partial}{\partial t} B^2}{\mu_0} - \frac{1}{2} \varepsilon_0 \frac{\partial}{\partial t} E^2 - \frac{\vec{\nabla} \cdot (\vec{E} \times \vec{B})}{\mu_0}$$

$$= -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0} + \frac{\varepsilon_0}{2} E^2 \right) - \vec{\nabla} \cdot \underbrace{\left(\vec{E} \times \vec{B}/\mu_0 \right)}_{\text{Give this term a name!}}$$

It's called $\vec{S}$ = "poynting vector" $= (\vec{E} \times \vec{B})/\mu_0$

Put it all together: Starting from p. 3 we have

$$\frac{dW}{dt} \equiv \iiint (\vec{E}\cdot\vec{J})\, d\tau$$

$$= -\frac{d}{dt}\underbrace{\iiint \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)d\tau}_{\text{Look back @ page 8.2, we called this } U_{\text{total},EM}} - \iiint (\vec{\nabla}\cdot\vec{S})\, d\tau$$

Defined on prev. page. Can use Divergence theorem!

$$\text{so } \underset{①}{\frac{dW}{dt}} = \underset{②}{-\frac{d}{dt}U_{EM}} - \underset{③}{\oiint \vec{S}\cdot d\vec{a}}$$

In words, ① = Work done on charges by EM fields

= ② Decrease in energy stored in the fields

minus ③ whatever energy flowed out the boundary.

Sign checks: If no energy flows (i.e. if ③ = 0), then

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt}$$ says increase of particle energy = loss of stored field energy

Makes sense, it's just energy conservation.

But if ③ ≠ 0, there's another mechanism to "feed" energy to particles

Apparently, $\vec{S}$ is the outflow of energy, so that a positive outflow yields a negative work on charges.

Summary: $\vec{S} = \dfrac{\text{Energy flow transported by } \vec{E} \,\&\, \vec{B}}{(\text{unit time})(\text{unit area})_\perp} = \dfrac{\vec{E}\times\vec{B}}{\mu_0}$

Locally (i.e. looking at densities rather than integrating over volume)

$$\underbrace{\frac{\partial U_q}{\partial t} = \vec{E}\cdot\vec{J}}_{\text{Bottom of p.3}} \underbrace{= -\frac{\partial}{\partial t}(u_{EM}) - \vec{\nabla}\cdot\vec{S}}_{\text{this is bottom of p.4}}$$

Poynting's Theorem 1884!

or, reorganizing,

$$\frac{\partial}{\partial t}(U_q + U_{em}) = -\vec{\nabla}\cdot\vec{S}$$

this is Griffiths' u_{mech}, particles' energy density.

this is the energy density of the E & B fields

this is the $\frac{\text{outflow}}{\text{volume}}$ of "energy current" with $\vec{S} = \vec{E}\times\vec{B}/\mu_0$

Might be complicated - certainly it's KE, but could also contain thermal or other forms of Potential energy.

The above is our classic, standard conservation law (c.f. page 3)

$$\frac{\partial}{\partial t}(\text{something}) = -\vec{\nabla}\cdot(\text{that something's associated current density})$$

$\vec{S}$ (poynting) is the energy current density $= \dfrac{\text{flow of energy}}{\text{sec}\cdot\text{m}^2}$

Compare this formula to:

$$\frac{\partial}{\partial t}(\rho) = -\vec{\nabla}\cdot\vec{J}$$

where $\vec{J} = \rho\vec{v} = \dfrac{\text{flow of charge}}{\text{sec}\cdot\text{m}^2}$

STRONG ANALOGY!

Globally (integrating over volume) we get back to

$$\frac{d}{dt}\underbrace{\iiint (u_q + u_{EM})\, d\tau} = \iiint -(\vec{\nabla}\cdot\vec{S})\, d\tau = -\underset{\updownarrow}{\oiint \vec{S}\cdot d\vec{a}}$$

rate of increase of all energy = − outflow of $\frac{\text{energy}}{\text{sec}}$

Local conservation laws teach us about "flow vectors"

Here, we have learned that $\boxed{\vec{S} \text{ is the } \frac{\text{energy flow vector}}{(\text{area})_\perp \cdot \text{sec}} = \frac{\vec{E}\times\vec{B}}{\mu_0}}$

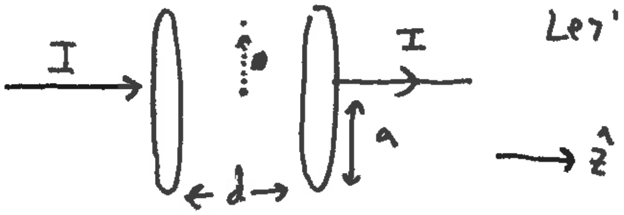
(& u_{EM} = stored energy density in fields) $= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}B^2/\mu_0$

Side Note: In MATERIALS, you can work out that

$$\vec{S} = \vec{E}\times\vec{H}$$

and $U_{em} = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$

<u>Example</u> :



Let's charge up a big capacitor (slowly ⇔ quasistatic)

<u>Inside</u>, $\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z}$ (By Gauss' law)

By Maxwell-Ampere, $\oint \vec{B}(s) \cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\varphi}$

around a circular "$\hat{\varphi}$ loop" of radius a

(Same result at $s=a$ if use $\mu_0\varepsilon_0 \frac{\partial E}{dt}$, check!)

So at edge of capacitor (at $s=a$)

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{Q}{A\varepsilon_0}\frac{\mu_0 I}{2\pi a}\underbrace{\hat{z}\times\hat{\varphi}} = -\hat{s}$$

energy is flowing <u>in</u> as we charge!

$$\frac{\text{Total energy out}}{\text{sec}} = \underbrace{\oiint}_{\substack{\text{over the cylindrical} \\ \text{outside of the capacitor}}} \vec{S}\cdot d\vec{a} = \frac{QI}{2\pi\varepsilon_0 a A}\cdot(-\hat{s})\underbrace{(2\pi a d\,\hat{s})}_{\text{outer area}}$$

$$= -\frac{QI}{\varepsilon_0}\frac{d}{A}.$$

thus, $\frac{\text{flow (in)}}{\text{second}} = +\frac{QI}{\varepsilon_0}\frac{d}{A}$

Now, U_E = stored energy $= \left(\frac{1}{2}\varepsilon_0 E^2\right)(\text{Volume}) = \frac{1}{2}\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2\cdot(A\cdot d)$

so $\frac{dU_E}{dt} = \frac{2Q}{2\varepsilon_0}\cdot\frac{dQ}{dt}\frac{d}{A}$ and, check it out

$= \frac{QI}{\varepsilon_0}\frac{d}{A}$ ⇐ increase of store energy /sec $=$ $\frac{\text{flow of energy in}}{\text{sec}}$

Nice! [* quasistatic ⇒ we're neglecting the tiny contribution from B-fields.]

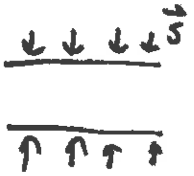
Example: consider a long wire, steady current flowing

Inside, from ch.7, we know $E = E_0\hat{z}$ and $\vec{J} = \sigma\vec{E} = \sigma E_0\hat{z}$

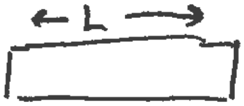
As usual, by Ampere (inside), $\vec{B} = \mu_0\frac{(J\cdot\pi r^2)}{2\pi r}\hat{\varphi} = \mu_0\frac{\sigma E_0}{2}r\hat{\varphi}$

At edge, $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{\sigma E_0^2}{2}a\,(\hat{z}\times\hat{\varphi}) \longrightarrow -\hat{s}$

Energy flow is (again) inwards



Now consider a piece of this wire of length L



across this, $\Delta V = E_0 \cdot L$ and $I = J\cdot\pi a^2 = \sigma E_0\pi a^2$

So $\frac{d}{dt}(W + U_{em}) = -\oiint\vec{S}\cdot d\vec{a}$ ← Poynting's theorem in integral form

Here, U_{em} is steady, so $dU_{em}/dt = 0$, and we get

$$\frac{dW}{dt} = -\oiint\vec{S}\cdot d\vec{a} = -\sigma\frac{E_0^2}{2}a\,(-\hat{s})\cdot(\underbrace{2\pi a L}_{\text{the outer area}}\hat{s})$$

the outer area: (end-caps contribute nothing!)

$$= +(\sigma E_0\pi a^2)(E_0 L)$$

$$= \underbrace{}_{I}\cdot\Delta V$$

Aha! Total power entering wire is $I\cdot\Delta V$, as we have always said! It enters via fields! Interesting: fields are all you need! Energy enters + is converted to U_{mech}, here in the form of thermal energy.

Momentum conservation : Recall (page 8.3) we started with energy, $dW_q = \vec{F}_{on\,q} \cdot d\vec{l}$. Let's instead start with force,

$$\vec{F}_{on\,q} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$ Newton's law.

Just as before, (like page 8.2), we can define a "momentum density"

$\vec{\mathbb{P}} = \frac{\text{momentum}}{\text{volume}}$ so that $\vec{p} = \iiint \vec{\mathbb{P}}\, d\tau$.

Then, locally (following the same general logic back on p. 3), we find

$$\frac{\partial \vec{\mathbb{P}}}{\partial t} = \rho \vec{E} + \vec{J} \times \vec{B}$$ (unlike p.3, the magnetic force doesn't disappear)

Again, as before, we could now eliminate ρ & $\vec{J}$ using Maxwell's eq'n. The vector algebra is a couple of pages of effort (Griffiths 351-3). Where does it take us? Just as w. the energy story, we find

$$\frac{\partial}{\partial t}(\vec{\mathbb{P}} + \vec{\mathbb{P}}_{fields}) = -\vec{\nabla} \cdot (\text{Something})$$

This is $\vec{\mathbb{P}}_{mech}$, defined above, it's $\frac{m\vec{v}}{\text{volume}}$

This is a vector involving $\vec{E} + \vec{B}$ fields. It turns out simple, $\vec{\mathbb{P}}_{fields} = \epsilon_0 \mu_0 \vec{S}$!

This requires the hard work. It must be OUR "momentum current" density

That "something" is curious. Note that $\vec{\nabla}\cdot$"something" is a *vector*!

What can *do* that? It's a *MATRIX*, a "2nd rank tensor".

It's named $\overleftrightarrow{T}$, the "stress-energy tensor".

Dotting with tensors is : $\vec{A}\cdot\overleftrightarrow{T} \equiv (A_x, A_y, A_z)\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$

= another vector!

+ similarly, $\vec{\nabla}\cdot\overleftrightarrow{T} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\begin{pmatrix} & & \\ & 3\times 3 & \\ & & \end{pmatrix} = \underset{\text{a vector!}}{(\quad,\quad,\quad)}$

$\overleftrightarrow{T}$ tells you $\dfrac{\overrightarrow{\text{momentum flow}}}{(\text{unit time})(\text{unit area})_\perp}$, transported by E & B

In *global* Form (integrating over a volume, like on p. 8-7)

$$\underbrace{\frac{d\vec{P}}{dt}}_{\substack{\text{Just } \vec{F}_{mech}, \\ \text{a vector}}} + \frac{d}{dt}\underbrace{\iiint \varepsilon_0 \mu_0 \vec{S}\, d\tau}_{\substack{\text{rate of change of our} \\ \text{"Stored momentum in} \\ \text{E + B fields", also a vector}}} = -\iiint(\vec{\nabla}\cdot\overleftrightarrow{T})\, d\tau$$

Divergence theorem!

$$= -\oiint \overleftrightarrow{T}\cdot d\vec{A}$$

3×3 matrix · vector

is *also* a vector.

This tells us the "inflow" (− sign!) of momentum through the boundary!

Summary: EM Fields store momentum. $\vec{P}_{EM} = \iiint \varepsilon_0 \mu_0 \vec{S}\, d\tau$

$= \iiint \varepsilon_0 \vec{E} \times \vec{B}\, d\tau$

Momentum density is $\vec{\mathbb{P}}_{em} = \varepsilon_0 \vec{E} \times \vec{B} = \varepsilon_0 \mu_0 \vec{S}$

$\overleftrightarrow{T}$ is $\dfrac{\text{outflow of } \overrightarrow{\text{momentum}}}{(\text{sec})\ (\text{area})_\perp}$

Note that in steady state situations, where $d\vec{P}_{EM}/dt = 0$,

the Force on some Volume with charges $= -\oiint \overleftrightarrow{T} \cdot d\vec{a}$ is given by this tensor out at the boundary!

Have a tensor (matrix) ⇒ some elements of $\overleftrightarrow{T}$ (the diagonal ones) act like pressures, since pressure · area = force.

But other elements (off diagonal) act like "shears",

e.g. you can generate an x-directed Force on a y-directed area element!

(Hence the name, "stress-energy tensor".)

It's a slight pain to derive, but fairly straightforward to compute, + is useful for calculating mechanical forces on charges in known E + B fields (-like in a plasma)

(We won't pursue it further, see Griffiths 8.2.2 for examples if you're curious!)

Angular momentum in fields:

Again, just as above, we can look at angular momentum of charged particles in fields, + compute a volume angular momentum density

$$\vec{\ell}_{EM} = \vec{r}\times\vec{\mathbb{P}}_{EM} = \varepsilon_0\,\vec{r}\times(\vec{E}\times\vec{B}) = \varepsilon_0\mu_0\,\vec{r}\times\vec{S}$$

- EM fields (even static ones) carry momentum + angular momentum!

Example

Solenoid, mounted on a platter, with a ring of λ. At $t=0$, all is at rest, + we close the switch.

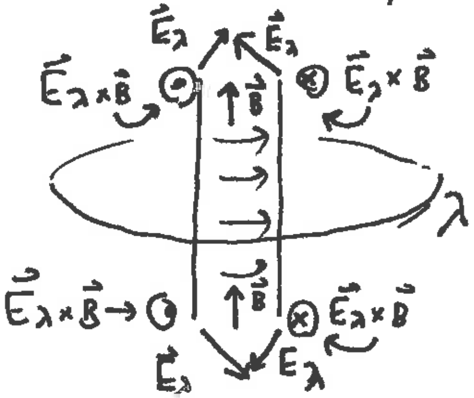
The solenoid ramps up, making a $\vec{B}_{inside}$ ↑↑↑, and Faraday says we get an induced $\vec{E}_{outside} \neq 0$, in the ↩ direction. So, the λ's feel a force, + the whole system starts to rotate!

Initially, $\vec{L}_{mech} = 0$, no $\vec{B} \Rightarrow \vec{\ell}_{EM} = 0$, ∴ NO angular momentum @ start

But after, $\vec{L}_{mech}$ is ↩ not zero. How can this conserve ∡ momentum?
↓$\vec{L}_{mech}$

In end, $\vec{B}\neq 0$, $\vec{E}\neq 0$, work it out, there's a nonzero $\vec{\ell}_{EM}$, upwards!

So $\vec{\ell}$ in fields compensates $\vec{L}_{mech}$, + angular momentum is conserved!



$\vec{r}\times(\vec{E}\times\vec{B})$ has an up component everywhere!

so $\vec{\ell}_{EM}$ is up, compensating $\vec{L}_{mech}$ down

Summary:

Conservation laws always say $\frac{\partial}{\partial t}(X) = -\vec{\nabla}\cdot\begin{pmatrix}\text{current associated}\\ \text{w. flow of } X\end{pmatrix}$

Charge: $\partial \rho/\partial t = -\vec{\nabla}\cdot\vec{J}$

Energy: $\frac{\partial}{\partial t}(u_{mech}+u_{Em}) = -\vec{\nabla}\cdot\vec{S}$ $\Big\}$ Fields have $\frac{\text{energy}}{\text{volume}} = \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right)$

$\vec{S}$ is the energy flow current.

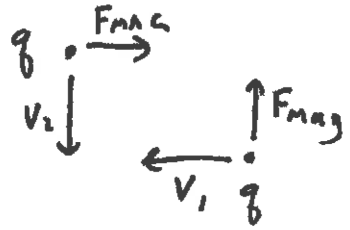
Momentum: $\frac{\partial}{\partial t}(\vec{\mathbb{P}}_{mech}+\vec{\mathbb{P}}_{em}) = -\vec{\nabla}\cdot\overleftrightarrow{T}$ $\Big\}$ Fields have $\frac{\overrightarrow{\text{momentum}}}{\text{vol}} = \mu_0\epsilon_0\vec{S}$

$\overleftrightarrow{T}$ is the stress-energy tensor, it's the $\overrightarrow{\text{momentum}}$ flow current.

4 momentum too, with $\frac{\text{ang momentum}}{\text{volume}} = \mu_0\epsilon_0\vec{r}\times\vec{S}$

Static fields can have $\vec{\mathbb{P}}$ and $\vec{\ell}$ in fields, "hidden momentum"

You must take this into account, if not, Newton's III law appears to fail: E.g.



← Looks like $\vec{F}_{12}\neq\vec{F}_{21}$, + momentum is thus not conserved. (Must compute $\vec{\mathbb{P}}_{Em}$, it's not zero!)

EM fields (think "photons"!) have no mass, but they carry E & $\vec{\mathbb{P}}$

For photons, $\frac{\text{Energy flow}}{\text{area}\cdot\text{time}}\cdot\frac{1}{c^2}$ = momentum density

$$|S|\cdot\frac{1}{c^2} = |\mathbb{P}_{em}|$$