3320 - 9.18

Energy & MOMENTUM IN WAVES:

Plane wave: $\vec{E}(\vec{r},t) = Re(\tilde{\vec{E}}(\vec{r},t)) = \vec{E}_0 \cos(\vec{K}\cdot\vec{r} - \omega t + \delta)$

Linearly polarized $\Rightarrow \vec{E}_0 = E_0 \hat{n}$, with $\hat{n}$ fixed and $\perp$ to $\vec{K}$

Also, $\vec{B} = \hat{k} \times \vec{E}/c$. (Since $\vec{E} \perp \vec{k}$, this means $|B_0| = |E_0|/c$)

Recall: u_{EM} = stored energy density $= \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2$. Easy enough!

$$u_{EM}^{\text{plane wave}} = \frac{1}{2}\varepsilon_0 |E_0|^2 \cos^2(\vec{K}\cdot\vec{r} - \omega t + \delta) + \frac{1}{2\mu_0}\frac{|E_0|^2}{c^2}\cos^2(\vec{K}\cdot\vec{r} - \omega t + \delta)$$

But $\mu_0 \varepsilon_0 = 1/c^2$, so $\frac{1}{2\mu_0 c^2} = \frac{1}{2}\varepsilon_0$, thus both terms are same.

In plane waves, $\vec{E}$ + $\vec{B}$ are "symmetric", storing equal energy!

$$u_{EM}^{\text{plane wave}} = \varepsilon_0 E_0^2 \cos^2(\vec{K}\cdot\vec{r} - \omega t + \delta) \leftarrow \text{A scalar quantity.}$$

This is ≥ 0, and it has a traveling wave pattern too.

If you average over time; $\langle \cos^2(\vec{K}\cdot\vec{r} - \omega t + \delta) \rangle = 1/2$, * Do you see why?

so $\langle u_{EM} \rangle = \frac{1}{2}\varepsilon_0 E_0^2$. Defined as $\frac{1}{T}\int_0^T u_E(\vec{r},t)\,dt$

Warning: In complex notation, if you try to compute a complex

$$\tilde{u}_{EM} \overset{???}{=} \frac{1}{2}\varepsilon_0 \tilde{E}_0^2 e^{2i(\vec{K}\cdot\vec{r} - \omega t + \delta)} + \frac{1}{2\mu_0}\tilde{B}_0^2 e^{2i(\vec{K}\cdot\vec{r} - \omega t + \delta)}$$

This fails: $Re(z^2) \neq (Re\, z)^2$

If you took $Re(\tilde{u}_{EM})$ you'd get $\cos(2(\vec{K}\cdot\vec{r} - \omega t) + \delta)$) → Not correct!

$Re(\tilde{u}_{EM}) \neq u_{EM}$!

3320 9.19

Poynting vector $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

Again, must be careful when using complex notation. That trick is linear, it fails for quadratic quantities like u or $\vec{S}$.

So do not try to compute $\hat{\vec{S}} \overset{???}{=} \frac{1}{\mu_0} \hat{\vec{E}} \times \hat{\vec{B}}$, that fails!

Use true $\vec{E}$ + $\vec{B}$ from previous page

$$\vec{S} = \frac{1}{\mu_0} \mathrm{Re}[\hat{\vec{E}}] \times \mathrm{Re}[\hat{\vec{B}}] = \frac{1}{\mu_0}(\vec{E}_0 \times \vec{B}_0) \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)$$

and $\vec{E}_0 \times \vec{B}_0 = \vec{E}_0 \times \left(\hat{k} \times \frac{\vec{E}_0}{c}\right) = \frac{E_0^2}{c} \hat{k}$ (Draw a picture, convince yourself!)

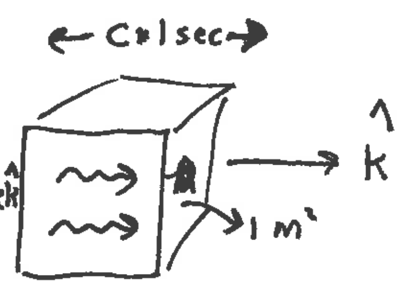
So $\vec{S} = \underbrace{\frac{E_0^2}{\mu_0 c}}_{} \cos^2(\vec{k}\cdot\vec{r} - \omega t + \delta)\, \hat{k}$. Use $c^2 = 1/\mu_0 \epsilon_0$

Can write as $\epsilon_0 c E_0^2$, or $\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0^2$, all are OK.

This is a flow of energy, moving in the $\hat{k}$ direction, as you'd expect

---

Note that $\vec{S} = u_{EM}\, c\, \hat{k}$. This makes sense too, consider a volume of space, 1 light-second "deep", all the EM energy stored in that box will flow out in 1 sec.



$$\frac{\text{Energy exiting}}{\text{per (second}\cdot\text{m}^2)} = u_{EM}\cdot c\hat{k} = \frac{\text{energy inside}}{\text{volume}} \times \text{volume}\, c\hat{k}$$

↳ This is $\vec{S}$, (just like $\vec{J} = \rho\vec{v}$, here $\vec{S} = u\vec{c}$)

3320 9.20

Time average $\langle \vec{S} \rangle = \underbrace{\frac{1}{2} \varepsilon_0 E_0^2 c}_{\text{From } \langle \cos^2 \rangle} \hat{k} = \langle u_{EM} \rangle c \hat{k}$

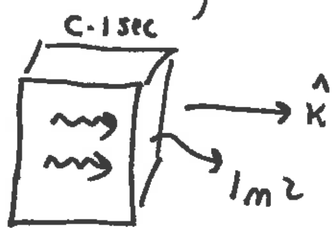
Finally, what about momentum density? Recall $\vec{\wp}_{EM} \equiv \mu_0 \varepsilon_0 \vec{S}$

this too is simple: $\langle \vec{\wp}_{EM} \rangle = \mu_0 \varepsilon_0 \langle u_{EM} \rangle c \hat{k} = \frac{\langle u_{EM} \rangle}{c} \hat{k}$

Interesting! Makes me think of the phys 2170 formula for relativistic momentum of photons, $p = \frac{E}{c}$!

If light hits matter & is absorbed, there will be a force.

Consider that box again:



Total $\vec{P}$ stored in this box is

$\vec{\wp} \times \text{volume} = \underbrace{\frac{\langle u_{EM} \rangle}{c} \hat{k}}_{\langle \wp_{EM} \rangle} \cdot \underbrace{c \cdot (1\text{sec} \cdot 1\text{m}^2)}_{\text{volume}}$. And in 1sec, all this momentum flows out, + gets absorbed.

So $\frac{\Delta \vec{P}_{\text{absorbed}}}{1\text{ sec}} = \frac{\langle u_{EM} \rangle}{c} \cdot c \cdot \frac{1\text{sec} \cdot 1\text{m}^2}{1\text{ sec}} \hat{k} = \langle u_{EM} \rangle \cdot 1\text{m}^2 \; \hat{k}$

this is $\vec{F}$, so $\frac{F}{\text{Area}} \equiv \text{radiation pressure} = \langle u_{EM} \rangle$

(If it were reflected, $\Delta \vec{p}$ would be twice as big, so would light pressure)

3320 9.21

Many times, in practice, we will be interested in

$$\frac{\langle \text{Power} \rangle \text{ transported across an area}}{\text{unit area}} \equiv \text{Intensity} \equiv I$$

this is $\langle \vec{S} \rangle \cdot \hat{k}$

so $I = \langle |S| \rangle = \langle u_{EM} \rangle \cdot c = \frac{1}{2}\varepsilon_0 c E^2 = P_{radiation} * c.$

This intensity goes like E^2, as you perhaps already knew.

Comments: In a medium, $I = \frac{1}{2}\varepsilon v E^2$, we'll come back to this

Everything here is classical. I invoke the word "photon" only suggestively! To really talk about photons, you're in a quantum paradigm. Energy is not "continuous" or arbitrary (or independent of ω!) as in the formulas above. This is all for classical EM radiation (lots of photons streaming by!)

3320 9.22

EM waves in matter : Till now we've considered waves in free space.

In matter, we must start with

$$\vec{\nabla}\cdot\vec{D} = \rho_F \qquad \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t \qquad \text{where} \quad \vec{D} = \varepsilon_0\vec{E} + \vec{\mathbb{P}}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \qquad \vec{H} = \frac{1}{\mu_0}\vec{B} - M$$

In analogy with our EM waves in empty space, let's consider EM waves in "EMPTY" matter, by which I mean $\rho_F = \vec{J}_F = 0$.

(so, uncharged glass, or insulator, or gas, or etc.)

The trick, as before, is to investigate the "curl of curl":

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

" Flyleaf !

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$$

Before, at this point, we invoked $\vec{\nabla}\cdot\vec{E} = 0$. But now... we don't know that

If $\vec{\nabla}\cdot\vec{\mathbb{P}} \neq 0$, then $\vec{\nabla}\cdot\vec{E} \neq 0$ when $\rho_F = 0$!

To keep life simple (for now), let's make two assumptions

1) Material is linear, so $\vec{D} = \varepsilon\vec{E}$
$\vec{H} = \vec{B}/\mu$

2) ε and μ are Homogeneous constants, properties of the material

(This means not a fn of position.)

3220 9.23

With those 2 simplifications, $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\left(\frac{\vec{D}}{\varepsilon}\right) = 0$ ← Since $\rho_F = 0$ + ε = constant

and $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\mu\vec{H}) = \mu\,\vec{\nabla}\times\vec{H}$ (Since μ is constant)

So we have $0 - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\underbrace{J_f}_{\text{Assumed } 0} + \partial\vec{D}/\partial t) = -\mu\,\partial^2\vec{D}/\partial t^2$

Thus, $\nabla^2\vec{E} = \mu\varepsilon\dfrac{\partial^2\vec{E}}{\partial t^2}$ (Just like our old friend $\nabla^2\vec{E} = \mu_0\varepsilon_0\,\partial^2\vec{E}/\partial t^2$)

A familiar wave equation

But the speed of this wave is $\frac{1}{\sqrt{\mu\varepsilon}}$, not $\frac{1}{\sqrt{\mu_0\varepsilon_0}} \equiv c$)

All our old results in vacuum follow in exactly the same way!

e.g. $\vec{B}\perp\vec{E}$, and both are $\perp\ \vec{K}$, and now $\vec{B} = \frac{1}{V}\vec{E}$

↳ rather than c!

Note that $\varepsilon > \varepsilon_0$, always ($\vec{E}$ fields polarize $\ominus \xrightarrow{\vec{E}} \oplus$ like this! $\Rightarrow \mathbb{P}$

It's true that μ can be $< \mu_0$, since $\xrightarrow{\vec{B}}$ $\Rightarrow\vec{M}$ or $\xrightarrow{\vec{B}}$ $\Leftarrow\vec{M}$

$\chi_M > 0,\ \mu > \mu_0$ Paramagnets

$\chi_M < 0,\ \mu < \mu_0$ Diamagnets

But, if less, it's usually only by parts per million!

Whereas ε is usually a lot $> \varepsilon_0$.

$V = \frac{1}{\sqrt{\mu\varepsilon}} \equiv \underset{\sim}{\frac{1}{n}}\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{c}{n}$, [where $n > 1$ for all known materials]

Defines "index of refraction" $\equiv n$.

3220 9.24

Conclusion: EM waves can propagate inside matter, but the EM waves are slower in matter (in "linear dielectrics")

The wave eq'n is nearly identical to free space, despite all the complicated physics, including quantum physics, inside insulators!

For glass, $\varepsilon \approx 2.25\,\varepsilon_0$, $\mu \approx \mu_0$, so $n = \sqrt{2.25} \approx 1.5$

For water, $\varepsilon \approx 80\,\varepsilon_0$, $\mu \approx \mu_0$ (But, this is for static E fields. Here, at high frequencies, we'll fine $n \neq \sqrt{80}$)

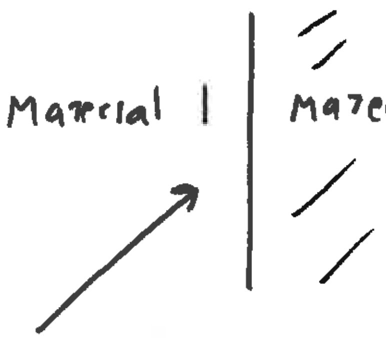
So for visible light, need to know the appropriate ε!

In glass, $\vec{E} + \vec{B}$ waves are polarizing the matter, creating dipoles, which themselves create a (time varying) $\vec{E}$ field from those dipoles, which superposes with the incoming $\vec{E}$.

The result is stunning: the frequency of the superposed wave is the same as what came in, but the speed is reduced!

3320 9-25

What about boundaries? This would mean ε is not homogeneous after all. We must solve our wave eq'ns in the two (separate, each homogeneous) media, + use BOUNDARY CONDITIONS to connect them!



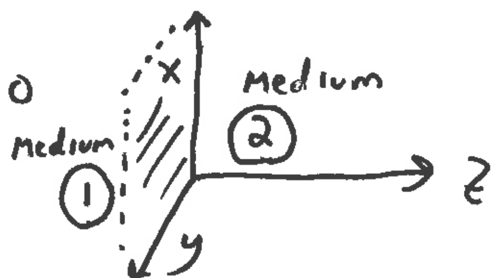
Plane wave comes in. What happens?

This is the "next easiest problem" after solving plane waves in a homogen. medium

[It's of great interest, think of optics!]

Conventions ① Assume monochromatic (single ω) plane waves. This simplifies the math, helps us make sense. But, plane waves are ∞ in extent, so it's not "physical". In the end, real problems demand Fourier summing these ideal plane waves to built real wave packets.

② We usually make our boundary $z=0$ (This is the xy plane)

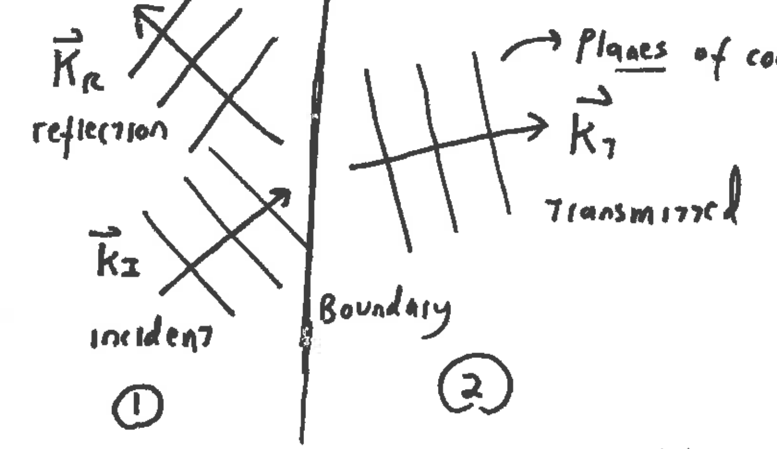


③ We usually assume incident waves come in from $-z$, heading right. Then Reflected wave superposes with incident in region ①

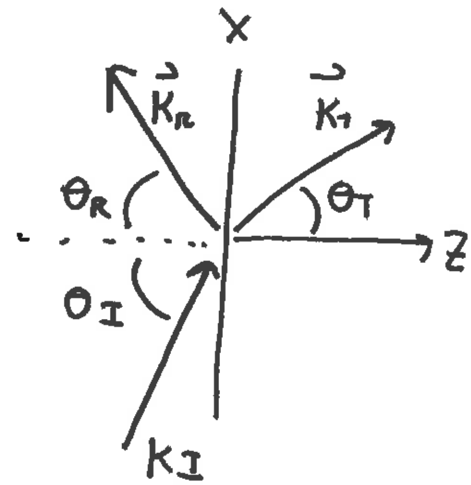
Transmitted " is all you have in region ②

④ Regions ① + ② are each homogeneous + linear.

3320 9-26



or, more simply



$\mu_1, \varepsilon_1, n_1$ $\qquad$ $\mu_2, \varepsilon_2\ n_2$ given

$V_1 = c/n_1$ $\qquad$ $V_2 = c/n_2$

we will apply Boundary conditions at $z=0$ plane, see below!

<u>Claims</u>:

$$\left.\begin{aligned} u &= \tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2\mu} B^2 \\ \vec{S} &= \tfrac{1}{\mu}\vec{E}\times\vec{B} \\ \text{INTENSITY}\quad I &= \tfrac{1}{2}\varepsilon V E_0^2 \\ \vec{B} &= \vec{E}/V \end{aligned}\right\}$$

(not ε_0, μ_0 in media!)

(not μ_0)

(not $\frac{1}{2}\varepsilon_0 c E_0^2$)

(not $1/c$)

Boundary conditions: Assume $\rho_F = \vec{J}_F\ (= \sigma_F = \vec{K}_F) = 0$!

From $\vec{\nabla}\cdot\vec{D} = \rho_F$ we get $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$

From $\vec{\nabla}\cdot\vec{B} = 0$ " " $B_1^{\perp} = B_2^{\perp}$

From $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ " " $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$

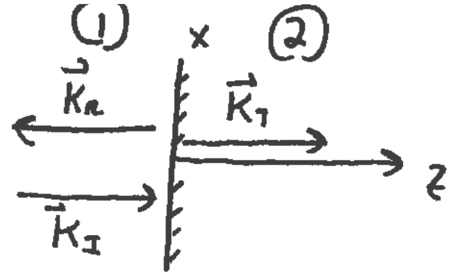
From $\vec{\nabla}\times\vec{H} = J_f + \frac{\partial\vec{D}}{\partial t}$ " " $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

Do you see why? Convince yourself, (what are the <u>tricks</u> to get these?) This is <u>very</u> general!

<u>These</u> conditions will tell us how light behaves at interfaces

Eyeglasses, microscopes, anti-reflection coating, it's all here!

3320 9-27



Case 1, normally incident light

INCOMING: $\vec{\tilde{E}}_I = \tilde{E}_{0I}\, e^{i(K_1 z - \omega t)}\, \hat{n}$ where $\hat{n} \cdot \hat{K} = 0$

If $\vec{E}_I$ is linearly polarized, $\hat{n}$ is any constant vector (in xy plane)

So let's define our x axis to be the polarization axis! This is still quite general, + lets us get specific right away

So our earlier results tell us, for incoming polarized plane wave:

$$\vec{\tilde{E}}_I = \tilde{E}_{0I}\, e^{i(K_1 z - \omega t)}\, \hat{x}$$

$$\vec{\tilde{B}}_I = \frac{\tilde{E}_{0I}}{v}\, e^{i(K_1 z - \omega t)}\, \hat{y}$$

This is required, comes from Maxwell Eq'ns in region (1)

This will hit the Boundary, producing Transmitted + reflected waves:

$\vec{\tilde{E}}_T = \tilde{E}_{0T}\, e^{i(K_2 z - \omega t)}\, \hat{n}_T$; $\vec{\tilde{B}}_T = \hat{K}_T \times \vec{\tilde{E}}_T / v_2$; $v_2 = \frac{\omega_T}{K_T}$

and $\vec{\tilde{E}}_R = \tilde{E}_{0R}\, e^{i(-K_1 z - \omega t)}\, \hat{n}_R$; etc. And $\frac{\omega_I}{K_I} = \frac{\omega_R}{K_R} = v_1$

Lots of unknowns! $\tilde{E}_{T0}$, $\tilde{E}_{R0}$, polarizations, frequencies, ...

you might guess $\hat{n}_T = \hat{n}_R = \hat{x}$ (why would polarization rotate?)

" " " $\omega_T = \omega_R = \omega_I$ (" " frequency change?)

$\boxed{\text{Boundary conditions will tell us!}}$

3320 9-28

B.C.: $\vec{E}^{\,\parallel}_{①} = \vec{E}^{\,\parallel}_{②}$. Here "//" means "parallel to the boundary"

But these waves are transverse, a normally incident wave is automatically entirely "parallel" to the boundary, (meaning direction of $\vec{E}$ is parallel)

$\therefore \vec{E}_{①} = \vec{E}_{②}$ at the $z=0$ boundary!

But by superposition, $\vec{E}_{①} = \vec{E}_I(z=0) + \vec{E}_R(z=0)$; $\vec{E}_{②} = \vec{E}_T(z=0)$

So $\tilde{E}_{0I}\, e^{i(0-\omega_I t)}\,\hat{x} + \tilde{E}_{0R}\, e^{i(0-\omega_R t)}\,\hat{n}_R = \tilde{E}_{0T}\, e^{-i\omega_T t}\,\hat{n}_T$

or, (constant vector) $e^{-i\omega_I t}$ + (constant vector) $e^{-i\omega_R t}$ = (constant vector) $e^{-i\omega_T t}$

I claim there's no way for such a relation to hold for all times, unless $\omega_I = \omega_R = \omega_T$. *(See supplemental p 28b for my proof)

(think about a simpler case, if $(\ )\cos\omega_1 t = (\ )\cos\omega_2 t$ for all t, this just cannot be true unless $\omega_1 = \omega_2$.)

So Boundaries don't change ω (as I suspected). This makes physical sense to me: $\vec{E}_I$ is driving oscillations at frequence ω_I + the material wiggles at that frequency, creating outgoing (reflected & transmitted) waves at that same frequency!

(Note that $v_1 \neq v_2$, so $k_1 \neq k_2$, wavelength is not same everywhere!)

3320 9-28b (supplement)

Suppose $Ae^{iat} + Be^{ibt} = Ce^{ict}$ for <u>all</u> t (and constant) $A, B, C \neq 0$

Can this be? Divide by e^{iat} to get

$$A + B^{i(b-a)t} = Ce^{i(c-a)t} \quad \text{for } \underline{\text{all}}\ t.$$

$$\left.\begin{array}{l} \text{At } t=0 \Rightarrow A+B=C \\ \text{At } t=\frac{2\pi}{b-a} \Rightarrow A+B = Ce^{i2\pi\frac{(b-a)}{(c-a)}} \end{array}\right] \text{ These must be } \underline{\text{equal}}.$$

So we <u>must</u> have $e^{i2\pi\frac{(b-a)}{(c-a)}} = 1$, meaning $c-a = b-a$, or $c=b$.

(you can think about how to extend this if you worry about $\frac{b-a}{c-a} = N \neq 1$)

So, apparently you need $b=c$, but <u>start</u> again with $b=c$

$$Ae^{iat} + Be^{ibt} = Ce^{ibt} \quad \text{for all } t.$$

$$\text{or} \quad A = (C-B)e^{i(b-a)t} \quad \text{" " "}$$

This is <u>Nonsense</u>! The left side is <u>constant</u>, the right side varies with time! Unless $b=a$, but then $b=a=c$!

(or, $C-B=A=0$, but I assume $A, B, C \neq 0$)

<u>Bottom line</u>: you cannot have 3 terms with <u>different</u> frequencies that always "cancel each other" or "balance" at <u>all</u> times, it can't happen!

3320 9-29

Back to our $\vec{E}_{①} = \vec{E}_{②}$ eq'n. (Now $e^{-i\omega t}$ cancels out, +

$$\tilde{E}_{0I}\,\hat{x} + \tilde{E}_{0R}\,\hat{n}_R = \tilde{E}_{0T}\,\hat{n}_T \qquad (\text{at } z=0) \qquad (1)$$

Next, consider another B.C. $\dfrac{\vec{B}_1^{\,\parallel}}{\mu_1} = \dfrac{\vec{B}_2^{\,\parallel}}{\mu_2}$ at $z=0$, which says

$$\frac{\tilde{E}_{0I}}{\mu_1 v_1}\,\hat{y} + \frac{\tilde{E}_{0R}}{\mu_1 v_1}(-\hat{k}_1 \times \hat{n}_R) = \frac{\tilde{E}_{0T}}{\mu_2 v_2}(\hat{k}_2 \times \hat{n}_T) \qquad (2)$$

(I used $\vec{B} = \dfrac{\hat{k}\times\vec{E}}{v}$ here, and noted $\vec{k}_R = -\vec{k}_I$, + cancelled $e^{-i\omega_2 t}$,

you must work this out! Assume $\left\{\begin{aligned} \hat{n}_R &= n_{Rx}\,\hat{x} + n_{Ry}\,\hat{y} \\ \hat{n}_T &= n_{Tx}\,\hat{x} + n_{Ty}\,\hat{y} \end{aligned}\right\}$ Fully general!

Work out those cross products, + then compare your eq'ns (1) & (2)

you will find they are inconsistent + impossible unless $n_{Ry} = n_{Ty} = 0$

Summary: using the "parallel" component boundary conditions, we learned ω's are all the same, & polarization doesn't rotate.

(This arose from linearity, there are ways to rotate polarization with nonlinear materials)

So our B.C (1) is now $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

B.C. (2) is $$\frac{\tilde{E}_{0I}}{\mu_1 v_1} - \frac{\tilde{E}_{0R}}{\mu_1 v_1} = \tilde{E}_{0T}/\mu_2 v_2$$

3320 9-30

we have 2 eq'ns in 2 unknowns, $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$. ($\tilde{E}_{0I}$ is given)

If you define $\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2}$, we have
$$\begin{cases} \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \\ \tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \end{cases}$$

- Normally, $\mu_1 \approx \mu_2 \approx \mu_0$, so $\beta \approx \frac{v_1}{v_2} = n_2/n_1$

so if we're going "low n to high n", like air to glass, $\beta > 1$

- Those 2 eq'ns are quick to solve, try it yourself!

$$\tilde{E}_{0T} = \frac{2}{1+\beta} \tilde{E}_{0I} \qquad \left(\approx \frac{2n_1}{n_1+n_2} \tilde{E}_{0I} \quad \text{if } \mu_1 \approx \mu_2 \approx \mu_0 \right)$$

$$\tilde{E}_{0R} = \frac{1-\beta}{1+\beta} \tilde{E}_{0I} \qquad \left(\approx \frac{n_1-n_2}{n_1+n_2} \tilde{E}_{0I} \quad \text{"} \right)$$

These are called Fresnel Equations. They tell us the reflected + transmitted waves. We're done! Given $\tilde{E}_I$, we have $\tilde{E}_R$ & $\tilde{E}_T$!
(Parallel) Boundary conditions totally solved the problem!

Notes: • n's are real, no complex phases are introduced.

- If $n_2 > n_1$, $\tilde{E}_{0R}$ has a sign flip, (but $\tilde{E}_{0T}$ never flips.)

Like wave on a string! Hitting a heavy rope (slow wave, bigger n) you will flip the reflected wave on the lighter string.

- If $n_1 = n_2$, $\tilde{E}_{0T} = \tilde{E}_{0I}$, $\tilde{E}_{0R} = 0$. (Good! $n_1 = n_2 \Rightarrow$ no boundary nothing happens!)

3320 9-31

What about energy flow, Intensity? Recall $I = \frac{1}{2} \varepsilon v E_0^2$

Define $T \equiv \frac{I_T}{I_I}$ = transmission coefficient $= \dfrac{\frac{1}{2}\varepsilon_2 v_2 E_{0T}^2}{\frac{1}{2}\varepsilon_1 v_1 E_{0I}^2} = \dfrac{\varepsilon_2 n_1}{\varepsilon_1 n_2} \dfrac{E_{0T}^2}{E_{0I}^2}$

Once more, assume $\mu_1 \approx \mu_2 \approx \mu_0$, so $n \equiv \sqrt{\dfrac{\varepsilon\mu}{\varepsilon_0 \mu_0}} \approx \sqrt{\varepsilon/\varepsilon_0}$

then $\varepsilon_1 \simeq \varepsilon_0 n_1^2$, and $\varepsilon_2 \simeq \varepsilon_0 n_2^2$, so $T = \dfrac{n_2}{n_1} E_{0T}^2 / E_{0I}^2$

From previous page, $T = \dfrac{n_2}{n_1} \cdot \dfrac{4 n_1^2}{(n_1+n_2)^2} = 4 n_1 n_2 / (n_1+n_2)^2$

Similarly, $R = \dfrac{I_R}{I_I} = \dfrac{E_{0R}^2}{E_{0I}^2} = \dfrac{(n_1-n_2)^2}{(n_1+n_2)^2}$

Notes: $R+T=1$, good, this is basically conservation of energy!

If $n_1 \approx n_2$, $T \to 1$, $R \to 0$ makes sense, "nothing happens"

- If either $n_1 \gg n_2$ or $n_2 \gg n_1$, $T \to 0$, $R \to 1$. "Mismatch $\Rightarrow$ poor transmission"

(See p. 9.31 b)

3320 9-31 b supplement #1

Some comments about R & T:

if $n_1 >> n_2$, we have $\tilde{E}_{0T} \simeq 2\tilde{E}_{0I}$ } This looks odd. We *fully* reflect, yet also transmit

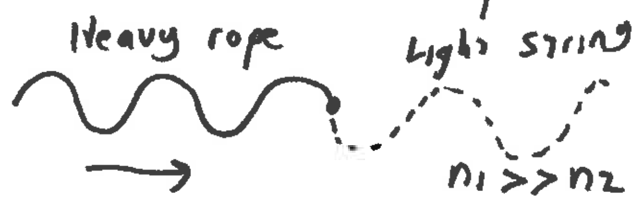
heavy glass → air $\tilde{E}_{0R} \simeq \tilde{E}_{0I}$ } Is this OK?

yes, because energy flow (T & R) involve velocity too (or n's)

As we saw, the much larger $V_2 \Leftrightarrow$ much smaller n_2, makes $T \to 0$!

Energy is "bunched up" in the big n_1 side, it's stored as polarization

A simple analogy might help:

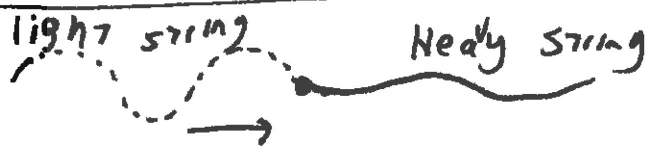


Yes, the light string has an amplitude, it wiggles plenty, but it is not carrying away energy, the energy all reflects!

Note in this case $E_{①}^{(z=0)} = E_0 + E_R = 2E_{0I} = E_{②}(z=0)$

so the amplitudes match up at b'dry in this case.

If reverse it,



$n_1 << n_2$

Now $E_T \to 0$, $E_R \approx -E_I$ (so @ $z=0$, no motion)

Again, $T \to 0$ here. In general, MISMATCH of n's at b'dry

⇒ "impedence mismatch" ⇒ poor transmission of energy

3320 9-31c supplement #2

In 1^{st} case of previous page $(n_1 >> n_2)$, we have, away from z

in ① $\tilde{\vec{E}}_{TOT} = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)} + \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}$

(use formulas we derived)

$$\simeq e^{-i\omega t}\left[\tilde{E}_{0I}\, e^{ik_1 z} + \tilde{E}_{0I}\, e^{-ik_1 z}\right]$$

$$= e^{-i\omega t}\, 2\tilde{E}_{0I} \cos k_1 z \quad .$$

If $\tilde{E}_{0I}$ is real,

$$Re\left(\tilde{E}_{TOT}\right) = 2E_{0I} \cos k_1 z \cos \omega t .$$

This is a standing wave. Energy $\rightarrow$ Energy $\leftarrow$ $\Big\}$ no actual net flow of energy, in ① or ②

In 2^{nd} case, with $n_1 << n_2$, we have $(z \neq 0)$

in ① $\tilde{\vec{E}}_{TOT} = \tilde{E}_{0I}\, e^{i(k_1 z - \omega t)} + \tilde{E}_{0R}\, e^{i(-k_1 z - \omega t)}$

$$= e^{-i\omega t}\left[\tilde{E}_{0I}\, e^{ik_1 z} - \tilde{E}_{0I}\, e^{-ik_1 z}\right]$$

$$= e^{-i\omega t}\, 2\tilde{E}_{0I}\, i \sin k_1 z \quad .$$

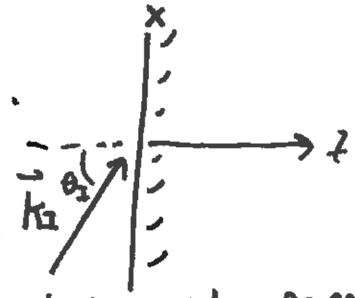
Again, if $\tilde{E}_{0I}$ is real,

$$Re\left(\tilde{\vec{E}}_{TOT}\right) = 2E_{0I} \sin k_1 z \sin \omega t$$

Again a standing wave!

3320 9-32

R & T for <u>oblique</u> incidence: Snell's Law!

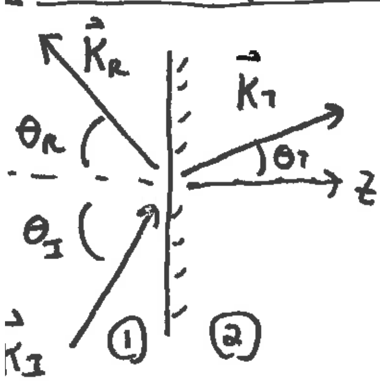


Let's consider incident waves entering at an angle. There is <u>some</u> plane defined by $\vec{K}_I$ and $\hat{z}$, it's called the "plane of incidence". (Here, it's the plane of this paper!)

Let's <u>define</u> our x-axis so the plane of incidence = xz plane = ".

The GAME is the same as our last example:

- Assume plane waves. Boundary <u>CONDITION</u> will tell us everything about reflected + transmitted waves. It's all just Maxwell's Eq'ns!



$$\vec{\tilde{E}}_I = \vec{\tilde{E}}_{0I}\, e^{i(\vec{K}_I\cdot\vec{r}-\omega t)} \quad ; \quad \vec{\tilde{B}}_I = \frac{\hat{K}_I \times \vec{\tilde{E}}_I}{v_1}$$

$$\vec{\tilde{E}}_R = \vec{\tilde{E}}_{0R}\, e^{i(\vec{K}_R\cdot\vec{r}-\omega t)} \quad ; \quad \vec{\tilde{B}}_R = \frac{\hat{K}_R \times \vec{\tilde{E}}_R}{v_1}$$

$$\vec{\tilde{E}}_T = \vec{\tilde{E}}_{0T}\, e^{i(\vec{K}_T\cdot\vec{r}-\omega t)} \quad ; \quad \vec{\tilde{B}}_T = \hat{K}_T \times \vec{\tilde{E}}_T / v_2$$

<u>As before</u>, <u>can't</u> get continuity of $\vec{E}$ & $\vec{B}$ unless ω's are all <u>same</u>.

Basically, if $(\text{Blah}_1)\, e^{i\omega_1 t} + (\text{Blah}_2)\, e^{i\omega_2 t} = (\text{Blah}_3)\, e^{i\omega_3 t}$ $(\forall t)$ you <u>must</u> have $\omega_1 = \omega_2 = \omega_3 = \omega$. No matter how complicated the "Blah's" are, if they don't depend on time, this is obligatory

Look at any of our B.C.'s (p. 9-26). They all result in some expression $(\text{Blah}_I)\, e^{i(\vec{K}_I\cdot\vec{r} - \omega t)} + (\text{Blah}_R)\, e^{i(\vec{K}_R\cdot\vec{r} - \omega t)} = (\text{Blah}_T)\, e^{i(\vec{K}_T\cdot\vec{r} -}$

The "Blah"'s are complex vectors, but they have no $\vec{r}$ or t dependences.

(Here, $\vec{r}$ will be any vector in the boundary plane $z=0$, i.e. $\vec{r} = (x, y, 0)$ the $e^{-i\omega t}$ cancels out.)

But same argument (see p. 286!) says, if $\vec{r}$ is arbitrary, then we must have $\underline{\vec{K}_I\cdot\vec{r} = \vec{K}_R\cdot\vec{r} = \vec{K}_T\cdot\vec{r}}$ for all $\vec{r}$ in $z=0$ plan[e]

Note that $\vec{K}_I = K_{Ix}\hat{x} + K_{Iz}\hat{z}$ is given, it's the incident wave.

so $(\vec{K}_I - \vec{K}_R)\cdot\vec{r} = 0$ for any $\vec{r} = (x, y, 0)$

i.e. $K_{Ix}\, x - (K_{Rx}\, x + K_{Ry}\, y) = 0$ for all x and all y.

That requires $K_{Ry} = 0$ $\leftrightarrow$ Reflected wave is also "in the page"!

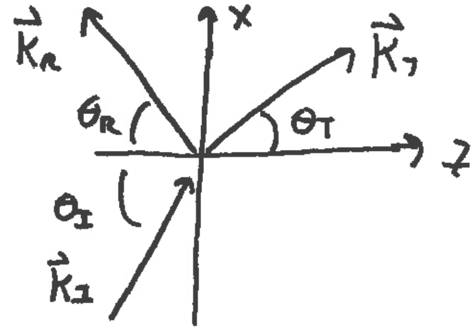
and $K_{Ix} = K_{Rx}$ $\leftarrow$ we'll examine this next page. *

Similarly $\vec{K}_I\cdot\vec{r} = \vec{K}_T\cdot\vec{r}$ tells us $K_{Ix}\, x - (K_{Tx}\, x + K_{Ty}\, y) = 0$

so here too, $K_{Ty} = 0$ $\leftrightarrow$ Transmitted wave is also "in the page"

and $K_{Ix} = K_{Tx}$ $\leftarrow$ we'll examine this next page. *

Don't forget: $K = \dfrac{\omega}{v}$ always, we'll use this



If $K_{Ix} = K_{Rx}$, then (see picture:)

$K_I \sin\theta_I = K_R \sin\theta_R$

But $K_I = K_R = \omega / v_1$, so

$\boxed{\sin\theta_I = \sin\theta_R.}$ $\angle$ of incidence $= \angle$ of reflection, all in one plan

This Law arises purely from Maxwell's equations!

Also, if $K_{Ix} = K_{Tx}$, then (see picture) $K_I \sin\theta_I = K_T \sin\theta_T$

But $\dfrac{K_T}{K_I} = \dfrac{\omega/v_2}{\omega/v_1} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$, so $\dfrac{\sin\theta_I}{\sin\theta_T} = \dfrac{K_T}{K_I} = \dfrac{v_1}{v_2} = \dfrac{n_2}{n_1}$

or $\boxed{n_1 \sin\theta_I = n_2 \sin\theta_T}$ Snell's law of refraction!

Again, a general law for any waves incident on a boundary.

We haven't explicitly used any of the B.C.'s in detail yet!

Snell's law arises only from assuming linear, homogeneous media.

(In fact, any waves that are "continuous" in any way will give this same result. It's quite general)

The above also tells us that when we do start applying our Maxwell Boundary Conditions in detail, that all $e^{i(\vec{K}\cdot\vec{r} - \omega t)}$ terms cancel!

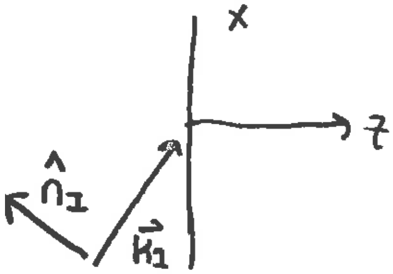
(So, for the next pages, we can cheerfully drop these for B.C.'s)

3320 9-35

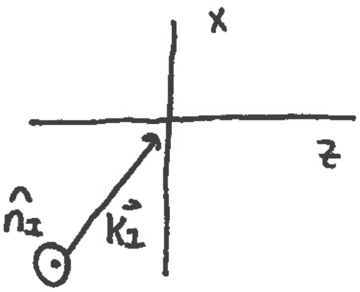
You might be content with Snell's law, but we can go further + find reflected + transmitted amplitudes + intensities too! To do this, though, we need to specify the polarization of $\vec{E}_I \equiv \hat{n}_I$. It matters!

Case 1) $\hat{n}_I$ lies in our plane of incidence ("the paper")

2) $\hat{n}_I$ is $\perp$ to our plane " " (")



Case 1)



case 2)

- Any other case is a linear combo of these!
- Results are similar but not identical. We'll do case 1), you can do case 2)!

ase 1) $\hat{n}_I$ has no $\hat{y}$ component: $\hat{k}_I = \sin\theta_I \,\hat{x} + \cos\theta_I \,\hat{z}$ ← convince youself!

$\hat{n}_I = \cos\theta_I \,\hat{x} - \sin\theta_I \,\hat{z}$ ← "

Quick check: $\hat{n}_I \cdot \hat{k}_I = 0$, as it must for travelling EM waves!)

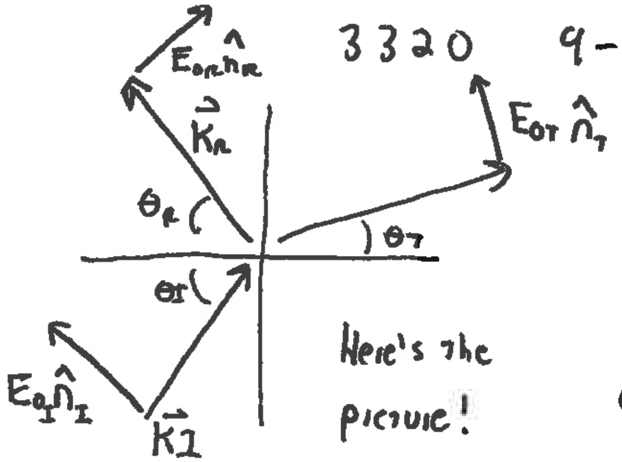
B.C.'s ee p.26!

$$\left\{ \begin{aligned} \varepsilon_1 E_{(1)}^{\perp} &= \varepsilon_2 E_{(2)}^{\perp} \\ B_{(1)}^{\perp} &= B_{(2)}^{\perp} \\ \vec{E}_{(1)}^{\parallel} &= \vec{E}_{(2)}^{\parallel} \\ \vec{B}_{(1)}^{\parallel} &= \vec{B}_{(2)}^{\parallel} \end{aligned} \right\}$$

- All of these are just for $z=0$
- "$\perp$" here means "$\perp$ to boundary", i.e. it means "take z-components"
- $B_1^{\perp} = B_2^{\perp}$ tells us nothing, since for case 1, neither $\vec{B}$ has any $B_\perp$! (convince yourself!)

(• Last eq'n turns out to be redundant, it adds nothing)

3320 9-36



Here's the picture!

1^{st} B.C., in $\hat{z}$ direction to get "$E^{\perp}$", says

$$\varepsilon_1 \left(\tilde{E}_{I0} (\hat{n}_I)_z + \tilde{E}_{0R} (\hat{n}_R)_z \right) = \varepsilon_2 \tilde{E}_{0T} (\hat{n}_T)$$

(I cancelled $e^{i(\ldots)}$ as promised)

Stare at the figure, take z components:

1^{st} B.C.: $\varepsilon_1 \left(\tilde{E}_{I0} (-\sin\theta_I) + \tilde{E}_{R0} (\sin\theta_R) \right) = \varepsilon_2 \tilde{E}_{T0} (-\sin\theta_T)$

Use Snell's law, $\sin\theta_I = \sin\theta_R$ and $\sin\theta_T = \sin\theta_I \, (n_1/n_2)$

$\Rightarrow \varepsilon_1 \left(\tilde{E}_{0I} (-1) + \tilde{E}_{0R} \right) = \varepsilon_2 \tilde{E}_{0T} (-1)(n_1/n_2)$

3^{rd} BC: $\vec{E}^{\parallel}_{①} = \vec{E}^{\parallel}_{②}$ Here, "$\parallel$" means "$\parallel$ to Boundary", i.e. "take x components", (since E has no y component

So $\tilde{E}_{0I} \cos\theta_I + \tilde{E}_{0R} \cos\theta_R = \tilde{E}_{0T} \cos\theta_T$.

(Convince yourself that the $\vec{B}^{\parallel}$ B.C. gives no new info!)

So we have 2 eq'ns in 2 unknowns $\tilde{E}_{0T}$ and $\tilde{E}_{0R}$:

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\varepsilon_2}{\varepsilon_1}\frac{n_1}{n_2} \tilde{E}_{0T} = \beta \tilde{E}_{0T}$$

(same β as on p. 30!)
$\beta \approx n_2/n_1$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \frac{\cos\theta_T}{\cos\theta_I} \equiv \alpha \tilde{E}_{0T}$$

(α is a constant. Given θ_I, Snell's law gives us θ_T, so it's fully determined by θ_I)

(These 2 eq'ns are easy to solve, do it yourself!)

These eq'ns give $\tilde{E}_{0T} = 2/(\alpha+\beta)\, \tilde{E}_{0I}$

$\tilde{E}_{0R} = (\alpha-\beta)/(\alpha+\beta)\, \tilde{E}_{0I}$

} Again called "Fresnel eqns"

That's it! We're done, we have the reflected + transmitted amplitudes!

Some observations: If $\theta_I = 0$, Snell $\Rightarrow$ $\theta_R = \theta_T = 0$, and $\alpha = 1$

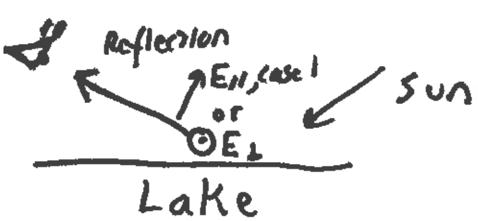
This restores our "normal incidence" formulas, a nice check.

$\tilde{E}_{0T}$ is always in phase with $\tilde{E}_{0I}$. The reflected wave can get a minus sign if $\alpha < \beta$, you need to look back at the picture on p. 36 to really interpret that sign, though! (Since $\hat{n}_R$ has x and z components)

But, no complex phases are introduced anywhere.

- The $\vec{B}$ fields all follow easily from $\hat{k} \times \vec{E} / v$.
- Unlike our "normal incidence" example, $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ do depend on θ_I, they are not simply determined by n_1 and n_2
- If $\alpha = \beta$, then $\tilde{E}_{0R} = 0$. That's interesting! This is a special angle, Brewster's angle, defined by $\alpha = \beta \Rightarrow \dfrac{\cos\theta_T}{\cos\theta_I} = \dfrac{n_1}{n_2}$

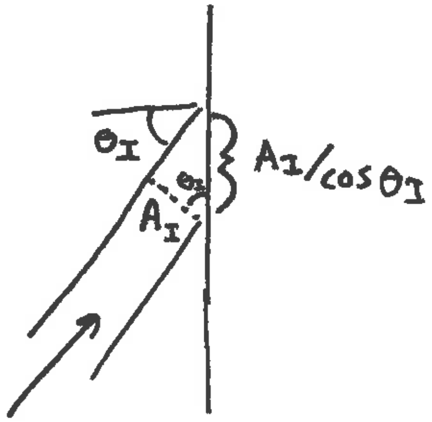
At this angle, all light transmits. (But remember, this is only true for "case 1", i.e. when polarization is in the plane.)



If sunglasses cut out polarization $\perp$ to plane, (e.g "vertically polarizing") then all glare at the Brewster angle will be 0, no reflection!

3320 9-38

For R & T (<u>Intensity</u> of Reflected & Transmitted light)



Geometrically:

$$|S| = \frac{\text{Power in}}{\text{Area}_{I,\perp}}, \text{ so Power} = |S| \cdot A_{I,\perp}$$

But <u>Intensity</u> is $\frac{\text{Power}}{\text{wall area}} = \frac{|S|\,A_{I,\perp}}{(A_{I\perp}/\cos\theta)}$

so $\underline{\underline{I = |S| \cos\theta}}$.

(Formally, $I = \langle \vec{S} \rangle \cdot \hat{z} \Rightarrow |S| \cos\theta$)

so $I_I = \langle S_I \rangle \cos\theta_I$, $I_R = \langle S_R \rangle \cos\theta_R$, $I_T = \langle S_T \rangle \cos\theta_T$

thus $R \equiv \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}$ (Recall $I = \frac{1}{2}\varepsilon v E^2$)

but T has a subtlety: $T \equiv \frac{I_T}{I_I} = \frac{n_2}{n_1} \frac{E_{0T}^2}{E_{0I}^2} \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta \cdot \frac{4}{(\alpha+\beta)^2}$

See notes p. 31, this comes from $\frac{1}{2}\underline{\underline{\varepsilon v}} E^2$

This is α, it comes from the geometry argument, above

so once again, $R + T = 1$ (as it must!)

See Griffiths for plots, (and the "trig tricks" to relate $\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$ to θ_I, using Snell's law)

9.38b

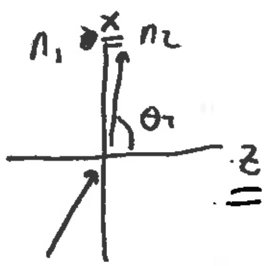
If $n_2 < n_1$ (e.g. water to air), there is a critical angle, θ_I (crit), $\sin\theta_c = n_2/n_1$, for which $\theta_T = 90°$. If $\theta_I > \theta_c$, you cannot transmit any light. You get Total Internal Reflection, TIR. This has many practical implications, e.g. low loss light transmission in optical fibers.

There's an interesting issue here, though. Suppose we try to use our formal results even when $\theta_I > \theta_c$. So $\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > 1$ (!!)

$$\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, e^{i(\vec{k}_2\cdot\vec{r}-\omega t)} = \tilde{\vec{E}}_{0T}\, e^{i(k_{2,x}\cdot x + k_{2,y}\cdot y + k_{2,z}\cdot z - \omega t)}$$

Here $k_2 = \frac{\omega}{c} n_2$ Formally, $k_{2,x} = k_2 \sin\theta_T = \frac{\omega}{c} n_2 \sin\theta_T = \frac{\omega}{c} n_1 \sin\theta_I$

Snell's law



$k_{2,y} = 0$

$$k_{2,z} = k_2\cos\theta_T = \frac{\omega n_2}{c}\underbrace{\sqrt{1-\sin^2\theta_T}}_{\text{Pure imaginary!!}}$$

Since $\sin\theta_T > 1$ here

So we have $\tilde{\vec{E}}_T = \tilde{\vec{E}}_{0T}\, \underbrace{e^{-\frac{\omega n_2}{c}\sqrt{\sin^2\theta_T - 1}\, z}}_{\text{Exponentially dies in } z \text{ direction.}}\, e^{i\omega\left(\frac{n_1 \sin\theta_I}{c} x - t\right)}$

So indeed, no energy flow in $+z$ direction, but still $E \neq 0$

This is called an "Evanescent Wave"

It's a classical effect, a bit reminiscent of quantum tunneling (there's no Poynting flux $\perp$ to surface, but physical effects are real & observable)