EM Waves in conductors: So far we've solved Maxwell's Eq'ns when $\rho = J = 0$ (in "free space" or linear dielectrics). We found SIMPLE TRAVELING WAVES (& then used BC's, Boundary Conditions, to find Reflection + Transmission, at the boundaries)

But what happens if the material conducts? The dominant physics is no longer "$\vec{E}$ interacting with bound charges", (although that is still present), it's "$\vec{E}$ interacting with free charges"!

As always, in full generality it can be complicated, but with a few simplifying assumptions we can get some cool results, including:

- Metals are shiny (R is generally big)
- Metals are opaque
- Metals have a small skin depth into which ($\vec{E}$) fields penetrate
- EM waves travel slower in metal than we might naively guess
- $\vec{B}$ & $\vec{E}$ are not in phase in metals ($\Rightarrow$ e.g. you get radiation pressure!)
- B dominates inside metals (most energy is in B fields)

All this comes from Maxwell's Eq'ns + BC's!

Statically, we know $\vec{E} = 0$ and $\rho_f = 0$ inside metals. But, what if $\vec{E}$ oscillates? Can you build up some charges locally?

Suppose you do: Imagine dumping $\rho_{f0} \neq 0$ deep inside a metal, at $t=0$. What happens? we know it will tend to repel itself + head to the edges. How long does that take?

$$\vec{\nabla}\cdot\vec{J} = -\partial\rho/\partial t \quad \leftarrow \text{This is conservation of charge, yes?!}$$

$$\vec{\nabla}\cdot(\sigma\vec{E}) = -\partial\rho/\partial t \quad \leftarrow \text{Ohm's law. } \sigma \equiv \text{conductivity, (it's not surface charge.)}$$

$$\sigma(\vec{\nabla}\cdot\vec{E}) = -\partial\rho/\partial t \quad \leftarrow \text{Let's assume homogeneous conductor}$$

$$\sigma\,\rho/\varepsilon_0 = -\partial\rho/\partial t \quad \leftarrow \text{That's Gauss' law!}$$

So $\partial\rho/\partial t = -\frac{\sigma}{\varepsilon_0}\rho$. This has a simple sol'n, $\underline{\rho = \rho_0 e^{-\frac{\sigma}{\varepsilon_0}t}}$

Conclusion: If you do build up a ρ_0 anywhere, it dissipates fast!

It's gone in time $\approx \tau_{Ohmic} = \varepsilon_0/\sigma = \frac{8.85\cdot 10^{-12} \text{ in SI units}}{10^8 \text{ for typical metals}} \approx 10^{-19}\,s$!!

For $t \gg \tau_{Ohm}$, we'll have $\boxed{\rho = 0 \text{ in conductors.}}$

So, no need to worry about charge buildup (except super high frequencies)

This seems reasonable, it's my strong intuition from Phys 3310, where we always said $\rho = 0$ everywhere inside metals.

If you take into account the fact that bound charges can polarize, just replace $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ with $\vec{\nabla}\cdot(\varepsilon\vec{E}) = \rho_F$

and on the right side, replace ρ with ρ_F, the free charges are the only ones that can "run away". This small change gives

$\tau_{\text{OHMIC, modified}} = \varepsilon/\sigma$ A slightly longer time. Makes sense to me, polarizing atoms in the lattice always reduces internal E fields, which are what are responsible for this dissipation of charge to surfaces.

There is yet another time scale involved here, it's hidden in $\vec{J} = \sigma\vec{E}$. Ohm's law arises from impurities & collisions, which have their own time scale! Typically, $\tau_{\text{collision}} \simeq \dfrac{\text{interaction distance}}{\text{typical } e^- \text{ speed}} \approx 10^{-14}\,s$ for normal metals

Bottom line: If we consider high frequency $\vec{E}$ fields then life will be more complicated. But if $f = \frac{1}{T} \ll \begin{cases} 1/\tau_{\text{ohmic}} \\ \text{and } 1/\tau_{\text{collison}} \end{cases}$ then

a) $f \ll 1/\tau_{\text{ohm}}$ lets us cheerfully set $\rho_F = 0$ in Gauss' law (because there's plenty of time for charge to dissipate)

b) $f \ll 1/\tau_{\text{collison}}$ lets us cheerfully use Ohm's law, $\vec{J} = \sigma\vec{E}$ (because there's plenty of time for collision to occur which give $\vec{J}$)

For $f \lesssim 10^{14}$ Hz, we should be good to go. This includes <u>radio</u>, TV, microwaves, ... al<u>most</u> up to visible light. (But, e.g X-rays really require a re-analysis!) Summarizing Maxwell's Eqn's:

$$\vec{\nabla} \cdot \vec{E} \simeq 0 \quad \longleftarrow \text{This is from prev page, } \rho \approx 0$$

$$\left.\begin{aligned} &\vec{\nabla} \cdot \vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t \end{aligned}\right\} \text{always!}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \partial \vec{D}/\partial t \Rightarrow \quad \text{(Assuming linear, homogeneous material)}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}_F + \mu \varepsilon \, \partial \vec{E}/\partial t$$

This set of eq'ns is nearly identical to our dielectric story (with $\rho_F = 0$ assumed) There is one small correction, we have an extra term $\mu \vec{J}_F = \mu \sigma \vec{E}$ (from Ohm's law). We <u>cannot</u> let $\vec{J}_f = 0$ here!!

Metals <u>conduct</u>, there <u>are</u> currents. This makes all the difference.

We will see that for good conductors (big σ) this new term <u>dominates</u>

So the final Maxwell eq'n is

$$\vec{\nabla} \times \vec{B} = \mu \sigma \vec{E} + \mu \varepsilon \, \partial \vec{E}/\partial t$$

$\hookrightarrow$ This is what's new. (It wasn't <u>there</u> for empty space, + we ignored any J_f in dielectrics too)

If we set up a situation like <u>last</u> section, i.e. sending EM waves <u>in</u> to a metal, this new term $\Rightarrow$ Ohmic currents appear. But these always dissipate <u>energy</u>. So we can not expect "free EM wave" sol'ns inside metals! Such waves should quickly dump energy into <u>thermal</u> heating. I expect $\vec{E}$ to dissipate rapidly with <u>distance</u> into metals. So I'm anticipating a "skin depth" effect, from Ohm's law.

Let's see! We use our (now standard!) trick to combine Maxwell Eqns, take the <u>curl</u> of Faraday's law, i.e. $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$

so $\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \vec{\nabla}^2\vec{E} = -(\mu\sigma\,\partial\vec{E}/\partial t + \mu\varepsilon\,\partial^2\vec{E}/\partial t^2)$

vanishes, as we argued

this comes from $-\frac{\partial}{\partial t}$ (Right side of Eq'n at bottom of previous page)

$$\text{so } \vec{\nabla}^2\vec{E} = \mu\sigma\,\partial\vec{E}/\partial t + \mu\varepsilon\,\partial^2\vec{E}/\partial t^2$$

<u>Similar</u> to our old wave eq'n!

This is <u>new</u>, and σ is big for good conductors, so it will matter!

Can we solve it? You can always just try something!

How about our usual ansatz, $\tilde{\vec{E}} = \tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$?

Let's just try it! ...

I'll pick $\hat{\vec{k}} \equiv \hat{z}$ for simplicity (pick your z axis!), so
$\vec{\tilde{E}} = \vec{\tilde{E}}_0 \, e^{i(kz-\omega t)}$. Taking $\vec{\nabla}^2$, only $\partial^2/\partial z^2$ will enter, +

∇^2 gives $-k^2$ out front. Similarly, $\partial/\partial t$ pulls down $-i\omega$,

and $\partial^2/\partial t^2$ gives $-\omega^2$ out front. So our eq'n reads

$$-k^2 \vec{\tilde{E}}_0 \, e^{i(kz-\omega t)} = -i\omega\mu\sigma \vec{\tilde{E}}_0 \, e^{i(\dots)} - \omega^2 \mu\varepsilon \vec{\tilde{E}}_0 \, e^{i\dots}$$

Cancelling out the common $\vec{\tilde{E}}_0 e^{i(\dots)}$, and a minus sign, we get

$$k^2 = \underbrace{i\omega\mu\sigma}_{\text{(From } \vec{J}_f)} + \underbrace{\omega^2\mu\varepsilon}_{\text{(From displacement current)}} . \quad \text{(complex!!)}$$

Conclusion: our trial sol'n does work, as long as k satisfies this.

(Let's rename it $\tilde{k}$, to remind ourselves this is now complex!)

The eq'n is familiar! In vacuum, where $\sigma = 0$, we used to have

$k^2 = \mu_0 \varepsilon_0 \omega^2$, which told us $v = \frac{k}{\omega} = 1/\sqrt{\mu_0 \varepsilon_0}$

Comment 1: $\vec{\nabla} \cdot \vec{E} = 0$ tells us $k(\vec{\tilde{E}}_0)_z = 0$, so once again, as before, our sol'n is transverse. $\vec{\tilde{E}}_0$ is $\perp$ to the $\hat{k}$ direction.

Comment 2: Taking curl of (Ampere-Maxwell) will, as usual, give us the exact same general sol'n for $\vec{B}$, with exact same $\tilde{k}^2$. Try it!

So we have a sol'n, except with the new twist that $\tilde{k}$ is complex.

we need $\vec{K}$, not $\hat{\vec{K}}^2$, in our sol'n $\hat{\vec{E}} = \hat{\vec{E}}_0 e^{i(\hat{K}z - \omega t)}$.

Taking the $\sqrt{\hat{K}^2}$ is slightly tricky, you need to think about it!

<u>Method 1</u>: Write $\hat{K}^2 = |\text{Magnitude}|\, e^{i\theta}$ (you can always write <u>any</u> complex # this way). Then $\hat{K} = \sqrt{|\text{Magn}|}\, e^{i\theta/2}$. Done!

<u>Method 2</u>: Write $\hat{K}^2 = a + bi$ (here $a = \mu\varepsilon\omega^2$, $b = \mu\sigma\omega$)

If $\hat{K} = K_R + K_{Im}\, i$, then $\hat{K}^2 = K_R^2 - K_{Im}^2 + 2K_R K_{Im}\, i$

This gives 2 eq'ns for 2 unknowns K_R + K_{Im}. Griffiths just writes down the sol'n,

$$K_R = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+(\sigma/\varepsilon\omega)^2}+1\right]^{1/2}$$

$$K_{Im} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+(\sigma/\varepsilon\omega)^2}-1\right]^{1/2}$$

(Try it yourself!)

But I'm going to simplify my life by continuing to assume we have a <u>good</u> conductor, i.e. σ is big. How big? Well, earlier we said we should limit ourselves to frequencies $<< \sigma/\varepsilon$ (to ensure no buildup of ρ anywhere in the conductor)

so by "good" I now mean $\boxed{\sigma >> \varepsilon\omega}$

and thus, $\sigma\mu\omega >> \mu\varepsilon\omega^2$, which means

$$\hat{K}^2 = \underbrace{i\sigma\mu\omega}_{\text{big } \sigma \text{ means this dominates}} + \underbrace{\omega^2\mu\varepsilon}_{\text{neglect}} \approx \underline{\underline{(\sigma\mu\omega)\, i}}$$

(In physics terms, $\vec{J}_f$ dominates over the displacement current!)

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We are limiting ourself to $\omega << \sigma/\varepsilon$, even for a "good" conductor. Numerically, $\sigma/\varepsilon \simeq 10^{19}$ or so, this leaves plenty of interesting physics to investigate. In this limit,

$$\hat{K} = \sqrt{\sigma\mu\omega}\,\sqrt{i} = \sqrt{\sigma\mu\omega}\,\sqrt{e^{i\pi/2}} = \sqrt{\frac{\sigma\mu\omega}{2}}\,(1+i)$$

(check! $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i$, Nice.) So $\hat{K} = K_R + K_{Im}\, i$

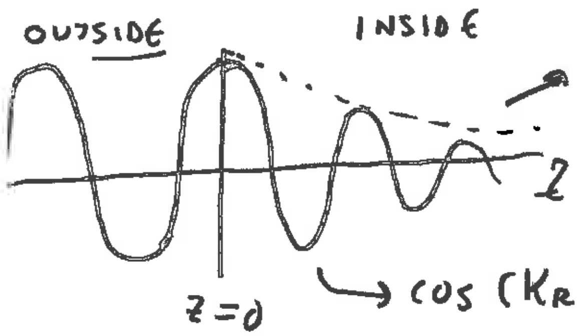
$$= \sqrt{\frac{\mu\sigma\omega}{2}} + \sqrt{\frac{\mu\sigma\omega}{2}}\, i$$

and our final sol'n for $\vec{E}$ in the conductor is

$$\vec{E} = \hat{\vec{E}}_0\, e^{i(\hat{K}z-\omega t)} = \hat{\vec{E}}_0\, e^{-K_{Im} z}\, e^{i(K_R z-\omega t)}$$

this is a constant, transverse vector out front * dying exponential * usual traveling wave, with speed $v = \omega/K_R$

(of course, as always, the physical $\vec{E}$ will be the real part of this!)



$e^{-K_{Im} z}$ envelope kills off our amplitude as we enter the metal.

$\cos(K_R z-\omega t)$ is the usual inward moving wave

This wave dies in distance $d \approx \frac{1}{K_{Im}} \equiv$ "skin depth"

In our "good conductor" case, $\underline{\underline{d \approx \sqrt{2/\mu\sigma\omega}}}$ (see above, where we found K_{Im})

Depends on frequency!

9-47

For normal metals with $\sigma \sim 10^{8}$, $\mu \approx \mu_0 \doteq 4\pi \cdot 10^{-7}$.

For radio waves, $\omega \sim 10^{6}$ or so (MHz), $d = \sqrt{\frac{2}{4\pi \cdot 10^{-7} \cdot 10^{8} \cdot 10^{6}}} \simeq 10^{-4}$ m

For optical, $\omega \sim 10^{15}$ or so (which is admittedly pushing our other assumption from collision time, but not by too much)

$d_{optical} \simeq \sqrt{\frac{2}{4\pi \cdot 10^{-7} \cdot 10^{8} \cdot 10^{15}}}$ ~ few nm, (10's of atomic distances)

Conclusions: Metal has small skin depth for any frequencies we'll likely to be interested in. So even a thin coat of metal will "act like a conductor" for EM waves. And, even a micron of Aluminum will block EM waves, so Aluminum foil is opaque to visible light!

Meanwhile, $K_R = \sqrt{\frac{\mu \sigma \omega}{2}}$, and the speed v of our travelling wave (called "v_p" for "phase velocity") is $\omega / K_R = \omega \sqrt{\frac{2}{\mu \sigma \omega}} = \sqrt{\frac{2\omega}{\mu \sigma}}$

Let's compare this to c: $\frac{v_p}{c} = \sqrt{\frac{2\omega}{\mu \sigma}} \cdot \sqrt{\mu_0 \epsilon_0} \approx \sqrt{2\omega \epsilon_0 / \sigma}$

But for "good conductors", $\sigma \gg \omega \epsilon$, so this means $v_p \ll c$.

- EM waves travel slower than c, by a lot. (Not because of ϵ!)

- Different frequencies have different speeds! (We'll come back to this, it's called "dispersion")

What about $\vec{B}$? I claimed $\vec{\nabla} \times$ (Maxwell-Ampere's law) gives

$\tilde{\vec{B}} = \hat{\vec{B}}_0 \, e^{i(\hat{K} z - wt)}$ (same general form, with same $\hat{K}$), + Faraday

$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ gives $i \, \hat{\vec{K}} \times \tilde{\vec{E}}_0 \, e^{i(\hat{K} z - wt)} = -(-i\omega)\, \tilde{\vec{B}}_0 \, e^{i(\cdots)}$

so $\hat{\vec{B}}_0 = \dfrac{\tilde{\vec{K}} \times \hat{\vec{E}}_0}{\omega}$. This is very familiar, except now $\hat{K}$ is complex!

We're used to $B_0 = E_0/v$, but now $|B_0| = \dfrac{|E_0|}{\omega/|\hat{K}|}$

Using $|\hat{K}| = \sqrt{K_R^2 + K_{Im}^2} = \sqrt{\mu \sigma \omega}$, we get $|B_0| = |E_0| \cdot \sqrt{\dfrac{\mu\sigma}{\omega}}$

$= \dfrac{|E_0|}{c} \sqrt{\dfrac{\sigma}{\varepsilon_0 \omega}}$

once again, note that $\sigma \gg \omega \varepsilon_0$ ("good conductor")

so B is much larger than you'd find in vacuum for that $\vec{E}$ field.

(And, correspondingly much more energy is stored in B^2)

The reason is that in vacuum, B arises purely from $\vec{J}_{displacement}$,

But now $\vec{J}_f$ dominates, its $\gg \vec{J}_{Disp}$, + so the $\vec{B}$ field comes

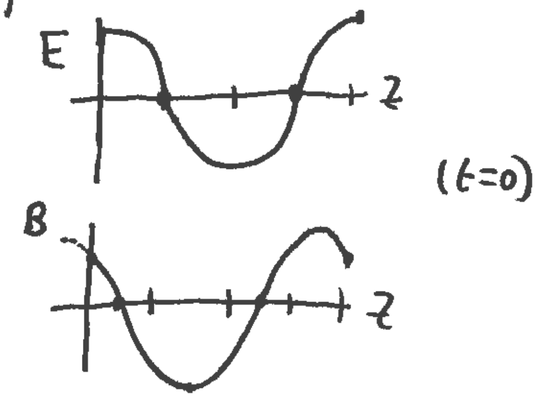
from the real physical currents produced by this $\vec{E}$)

Since $\vec{B}_0 = \frac{\tilde{\vec{k}}}{\omega} \times \vec{E}_0$ with $\tilde{k}$ complex, there is a <u>phase</u> difference between $\vec{B}$ & $\vec{E}$ that we've never had up to now. In our "good conductor" limit, $\tilde{k} = \sqrt{\mu\sigma\omega}\left(\frac{1+i}{\sqrt{2}}\right) = \sqrt{\mu\sigma\omega}\, e^{i\pi/4}$

So $\vec{B}$ is shifted by $\pi/4$ ($1/8$ of a cycle) from $\vec{E}$.

So if $\vec{E} = \ldots \cos(k_R z - \omega t)$ then

$\vec{B} = \ldots \cos(k_R z - \omega t + \frac{\pi}{4})$



$\vec{B}$ <u>lags</u> behind $\vec{E}$, by $1/8$ of a cycle.

This phase shift has an interesting physical consequence.
$\vec{E}$ drives free electrons, they <u>move</u> in $\hat{E}$ direction, + as they move, $q\vec{v}\times\vec{B}$ applies a Lorenz force, in the $\hat{v}\times\hat{B} = \hat{S}$ or $\hat{k}$ direction

If $\vec{E}$ + $\vec{B}$ were perfectly in phase, $\vec{v}$ & $\vec{B}$ would be $90°$ out of phase (By newton's law! $\vec{F} = q\vec{E} = m\dot{\vec{v}}$, and one derivative brings down an i from $e^{i\omega t}$, which is $90°$). Thus, you'd get <u>no</u> time average Lorentz force. But <u>with</u> the lag, you get a nonzero $\langle \vec{F}_{Lorentz} \rangle$

So this is a <u>physical</u> origin of radiation pressure!

9.49b (Details)

<u>Lorentz force</u>:

If $\vec{E} = E_0 \hat{x} e^{i(kz - \omega t)}$, + $q\vec{E} = m\ddot{\vec{x}}$, go to $z=0$, try $x = x_0 e^{-i\omega t} \hat{x}$

so $qE_0 = -m\omega^2 x_0 \Rightarrow x_0 = \dfrac{qE_0}{m\omega^2} e^{i\pi}$

and $\vec{v} = \dot{\vec{x}} = -i\omega\vec{x} \Rightarrow v_0 = \dfrac{qE_0}{m\omega^2} e^{i\pi/2}$

Also $\vec{B} = \dfrac{E_0}{v} e^{i\pi/4} \hat{y} e^{-i\omega t}$ ($e^{i\pi/4}$ is explained in notes, for good conductor)

$$\Rightarrow B_0 = \frac{E_0}{v} e^{i\pi/4}$$

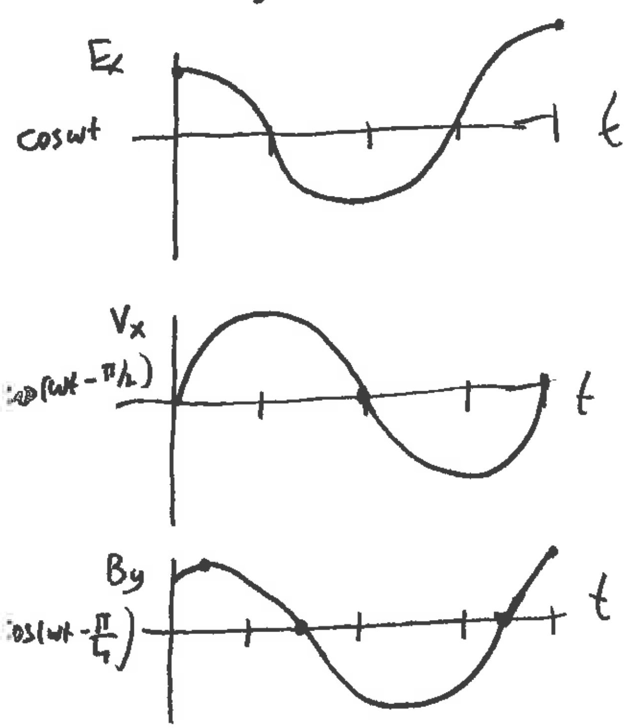
$$\vec{F}_{mag} = q\vec{v}_{real} \times \vec{B}_{real} = \frac{q^2 E_0^2}{m\omega v} \cos(kz - \omega t + \pi/2)\cos(kz - \omega t + \pi/4)\ \hat{z}$$

Note $\langle \cos(\omega t - \pi/2)\cos(\omega t - \pi/4)\rangle = \langle \sin\omega t \ \frac{1}{\sqrt{2}}(\cos\omega t + \sin\omega t)\rangle$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$$

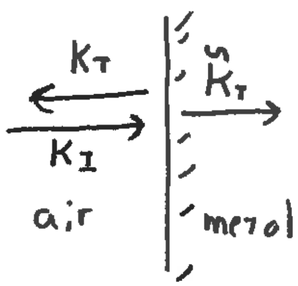
Had there <u>been</u> no $\pi/4$, we'd have $\langle \sin\omega t \cdot \cos\omega t\rangle = 0$!!

So $\langle \vec{F}_{mag}\rangle$ is in $+\hat{z}$ direction!



[redacted],

Reflection off metals: Let's consider Normal incidence of an EM wave on a conductor (for simplicity!)



Boundary conditions:

$$\begin{cases} \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \\ B_1^{\perp} - B_2^{\perp} = 0 \\ \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \\ \vec{B}_1^{\parallel}/\mu_1 = \vec{B}_2^{\parallel}/\mu_2 \end{cases}$$

Fortunately, irrelevant for normal incidence, so σ_f won't matter

← If no surface currents, see below.*

Our BC's here turn out to look the same as when we solved for normal incidence on a dielectric! So the result will carry over,

$$\tilde{E}_{0R} = \tilde{E}_{0I}\left(\frac{n_1 - n_2}{n_1 + n_2}\right) \quad \left(\begin{array}{l}\text{we will need to think about how to} \\ \text{define } n_2 \text{ now, it will be complex!}\end{array}\right)^{**}$$

* What about $\vec{K}_f$ on surface? Griffiths points out that for an Ohmic metal, $\vec{J} = \sigma\vec{E}$ says to get a "delta fn" (infinite) surface current you'd need an infinite $\vec{E}$ there, which is unphysical. You get $\vec{J}$ inside arising from $\vec{E}$, but no singular ($\vec{K}$) currents at the edge)

** Before, $n \equiv \dfrac{ck}{\omega}$, so to use this formula, $\tilde{n}_2 = \dfrac{c\tilde{k}_2}{\omega}$ will be complex.

This means there are non-trivial phase relations between $\tilde{E}_{0R}$ & $\tilde{E}_{0I}$ now!

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In our "good conductor" limit, $\hat{k}_2 = k_R + i\,k_{Im} = \sqrt{\frac{\mu\sigma\omega}{2}}\,(1+i)$

whereas k_1 is real, it's just ω/c, and recall (p. 47) $k_R \gg \frac{\omega}{c} = k_1$
$= k_{Im}$

so $$\tilde{E}_{0R} = \tilde{E}_{0I}\left(\frac{k_1 - \hat{k}_2}{k_1 + \hat{k}_2}\right) = \left(\frac{(k_1 - k_R) - i\,k_{Im}}{(k_1 + k_R) + i\,k_{Im}}\right)\hat{E}_{0I}$$

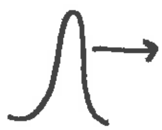
(k_1 is tiny in the good conductor limit) so $\hat{E}_{0R} \approx -\tilde{E}_{0I}$

Similarly, $R = \frac{|E_{0R}|^2}{|E_{0I}|^2} \approx 1$.

So we get essentially perfect reflection from good conductors. (Hence, shiny metals!) Since d is small, even a very thin coating of conductor is shiny - this is how you make a mirror!

(I'd say the conduction electrons in the metal cancel out the $\vec{E}$ field inside, + thus $\vec{E} \to 0$ at the boundary, so that's physically why $\vec{E}_{0R}$ must cancel out $\vec{E}_{0I}$ at the boundary)

<u>Dispersion</u> : We noted that in metals, $|K|$ depends on ω. So the <u>speed</u> of a plane wave depends on its frequency. Different <u>colors</u> behave differently! And if you build up a localized packet using Fourier's idea, the different ω's in the sum travel at different speeds.

So a wave packet  will <u>spread out</u> spatially! The higher ω components will travel at a different speed, thus "dispersing" this packet over time. Hence the name "dispersive medium"

This physics yields rainbows + prisms (different "n" for different ω)

- Conductors are dispersive (we found $K \sim \sqrt{\omega}$) but they also kill off the amplitude, so you don't <u>see</u> this effect as readily.
- Dielectrics are dispersive too. We treated them as linear in ω (with a constant ε (+ thus n) independent of ω) for simplicity, but this turns out not to be true for real dielectrics.

Let's return to dielectrics, + <u>model</u> the interaction of EM waves with molecules to <u>derive</u> this dispersion, i.e. find $\omega(K)$

Our model will be crude + classical, just to get a sense for the physics!

Preliminary comments: Fourier tells us any "wave packet"

$$f(x,t) = \int a(k)\, e^{i(kx-\omega t)}\, dk .$$

traveling wave

= Sum of plane waves of different k.

[Without dispersion, $\frac{\omega}{k} = v$ is a constant for all k. With dispersion, $\omega = \omega(k)$ need not be linear!]

The details here are tricky, I don't want to dig in too deeply. Here are some key takeaways:

1) If you build a localized traveling packet, you need multiple k's. But if these are "concentrated" around some dominant central $k_0 = \omega_0/v_0$ then it turns out the wave packet's "center" travels at a speed

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$$

. This is the "group velocity", as versus the "phase velocity" $v_p = \omega/k$ of the plane waves

2) Relativity insists $v_g < c$ for any physical waves.

Information (+ energy) travels at v_{group}.

(you <u>can</u> have $v_p > c$ in some cases, but <u>never</u> find $v_g > c$!)

For EM waves in matter, we want to know the "dispersion equation" $\omega(k)$ (or equivalently $k(\omega)$), so we can <u>deduce</u> $\frac{d\omega}{dk}$ + thus the speed of travel of information.

9-53 b. More comments on V_g + V_p.

In AM radio, the "carrier signal" is a high frequency plane wave.

It's effectively everywhere at all times, $e^{i(\vec{K}\cdot\vec{r}-wt)}$ with $w = cK$

It carries no information (except "I'm there")

But if you modulate the amplitude in a limited region of space + time, you have a signal. That signal (the music) travels at speed V_g.

In vacuum, $V_g = c$ too, but in dispersive medium (like air), $V_g < c$)

(9.53c)

$$f(x) = \int_{-\infty}^{\infty} a(K)\, e^{iKx}\, dK$$ Consider a "mostly K_0" colored pulse:

Let $\boxed{a(K) = \sqrt{\frac{\sigma}{\pi}}\, e^{-\sigma(K-K_0)^2}}$, then $f(x) = \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(K-K_0)^2} e^{iKx}\, dK$ Let $K' = K - K_0$

$$= \sqrt{\frac{\sigma}{\pi}} \int_{-\infty}^{\infty} e^{-\sigma(K')^2} e^{iK'x} e^{iK_0x}\, dK' = \sqrt{\frac{\sigma}{\pi}} \int e^{-\sigma\left(K' - \frac{ix}{2\sigma}\right)^2} e^{-x^2/4\sigma} e^{iK_0x}\, dK'$$

$$= \sqrt{\frac{\sigma}{\pi}}\, e^{-x^2/4\sigma} e^{iK_0x} \underbrace{\int e^{-\sigma(K'')^2}\, dK''}_{\sqrt{\pi/\sigma}} = \boxed{e^{-x^2/4\sigma} e^{iK_0x} = f(x)}$$

[This is localized in x (but with a phase that depends on the primary color]

Now, what is $\int_{-\infty}^{\infty} a(K)\, e^{i(Kx - \omega t)}\, dK$, (If $\omega = V_p K$) ← If in free space, no dispersion

$$= \int_{-\infty}^{\infty} a(K)\, e^{iK(x - V_p t)}\, dK = f(x - V_p t)$$ ← As expected, pulse moves at V_p.

What is $\int_{-\infty}^{\infty} a(K)\, e^{i(Kx - \omega t)}\, dK$, if $\omega = \omega(K) \approx \omega(K_0) + (K - K_0) \left.\frac{d\omega}{dK}\right|_{K_0} + \ldots$

Can't do the integral unless linearize

$\equiv V_p K_0 + (K - K_0) V_g + \ldots$

(Center on K_0, since that's where $a(K)$ "lives")

$$= \int_{-\infty}^{\infty} \underbrace{\sqrt{\frac{\sigma}{\pi}}\, e^{-\sigma(K-K_0)^2}}_{a(K)} e^{i(Kx - V_p K_0 t - (K-K_0)V_g t)}\, dK$$

write $K = K - K_0 + K_0$

$$= \int_{-\infty}^{\infty} a(K - K_0 + K_0)\, e^{i(K-K_0)(x - V_g t)} e^{-iV_p K_0 t} e^{iK_0(x)}\, dK$$

$$= e^{-iV_p K_0 t} e^{iK_0(x)} \int_{-\infty}^{\infty} a(K - K_0 + K_0)\, e^{i(K-K_0)(x - V_g t)}\, d(K - K_0)$$

$$= e^{iK_0(x - V_p t)} \int_{-\infty}^{\infty} a(K' + K_0)\, e^{iK'(x - V_g t)}\, dK'$$

$K'' = K' + K_0$

Note $\int a(K' + K_0)\, e^{iK'x}\, dK' = \int a(K'')\, e^{iK''x}\, dK''\, e^{-iK_0x} = e^{-iK_0x} f(x)$

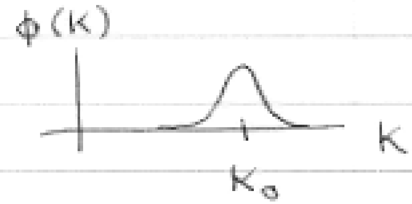
So we have $e^{iK_0(x - V_p t)} \left[e^{-iK_0(x - V_g t)} f(x - V_g t) \right] = e^{iK_0(V_g - V_p)t} f(x - V_g t)$

Uninteresting phase.

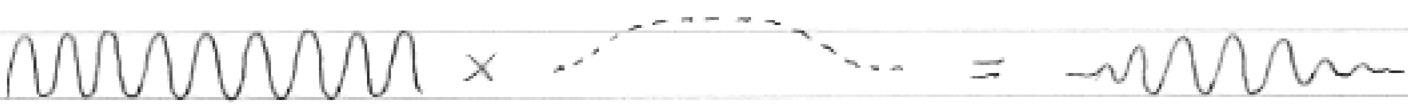
Result: Travels @ V_g, not V_p!

Back to problem of velocity of free particle

Wave Packet: 



Will show that ripples inside packet move w/ phase velocity ω/k but envelope  moves w/ group velocity $d\omega/dk$



ripples × envelope = packet

packet $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(K)\, e^{i(Kx - \omega(K)\cdot t)}\, dK$

$$\omega = \omega(K) = \frac{E}{\hbar} = \frac{\hbar k^2}{2m}$$

$\phi(K)$ is ~~xx~~ assumed $\simeq 0$ except near $K_0 = 2\pi/\lambda_0$

$\Rightarrow$ only contributions to integral from K's near K_0

$\Rightarrow$ can expand $\omega(K)$ about K_0

Taylor Series: $\omega(K) = \omega(K_0) + \left.\frac{d\omega}{dk}\right|_{K_0} \cdot (K - K_0)$

$$\omega(K) = \omega_0 + \omega_0' \cdot \Delta K$$

$\Delta K = K - K_0$, $\quad K = K_0 + \Delta K$,

$$e^{i(Kx - \omega t)} = e^{i[(K_0 + \Delta K)x - (\omega_0 + \omega_0' \cdot \Delta K)\cdot t]}$$

$$= e^{i(K_0 x - \omega_0 t)} \cdot e^{i(\Delta K \cdot x - \omega_0' \Delta K t)}$$

$$= e^{i(K_0 x - \omega_0 t)} \cdot e^{i \Delta K (x - \omega_0' t)}$$

So, can rewrite $\Psi(x,t)$ as

$$\Psi(x,t) = \underbrace{\frac{e^{i(k_0 x - \omega_0 t)}}{\sqrt{2\pi}}}_{\substack{g\left(x - \frac{\omega_0}{k_0} t\right) \\ \text{(ripples)}}} \underbrace{\int_{-\infty}^{+\infty} d(\Delta k)\, \phi(k_0 + \Delta k)\, e^{i \Delta k (x - \omega_0' t)}}_{\substack{f(x - \omega_0' \cdot t) \\ \text{(envelope)}}}$$

phase velocity $v_{phase} = \dfrac{\omega}{k} = \dfrac{\hbar k}{2m} = \dfrac{p}{2m}$

(recall $\omega = \omega(k) = \hbar k^2/2m$)

group velocity $v_{group} = \dfrac{d\omega}{dk} = \dfrac{\cancel{2}\hbar k}{\cancel{2}m} = \dfrac{\hbar k}{m} = \dfrac{p}{m}$

$\Rightarrow$ envelope of wavepacket moves w/ $v_{group} = p/m$ agrees w/ classical mechanics

Because different k's move w/ different $v_{phase} = \dfrac{\hbar k}{2m}$ the envelope tends to spread out as it moves w/ higher k's moving to front of packet

Δx grows. OK w/ Uncertainty Principle which only places lower limit $\Delta x \Delta p \geq \hbar/2$

Our simple model will be "atoms are charges on springs," 

So in an $\vec{E}$ field, Newton says $\vec{F}_{\substack{net \\ on\ q}} = q\vec{E} - K_{spring}\vec{x} - (drag)\cdot\vec{v}$

$K_{spring} = m\omega_0^2$ (to avoid confusing "spring K" + wavenumber)

ω_0 is the natural (resonant) frequency of the charge.

drag $\equiv m\gamma$ is some internal friction (e.g. from radiation, which we'll get to soon). Let's let $\vec{E} = E_0\hat{x}e^{i(kz-\omega t)}$

Focus on any one atom, take x components, go to $z=0$, + you have

$$m\ddot{x} = \underbrace{qE_0e^{-i\omega t}}_{\text{the "driver"}} - \underbrace{m\omega_0^2x}_{\text{the spring}} - \underbrace{m\gamma\dot{x}}_{\text{the damping}}$$

where $x(t)$ is the displacement of our charge q in the atom

This is a familiar ODE, (see phys2210!), just try a sol'n $x = x_0e^{-i\omega t}$

$$-m\omega^2x_0 = qE_0 - m\omega_0^2x_0 - m\gamma(-i\omega)x_0 \quad \text{(After canceling } e^{-i\omega t}\text{)}$$

which is solved by the right choice for x_0,

$$x_0 = qE_0/m \Big/ (\omega_0^2 - \omega^2 - i\gamma\omega)$$

This means the atom polarizes, $p(t) = \underbrace{q\,x(t)}_{\text{usual dipole formula}} = q\,\underbrace{x_0e^{-i\omega t}}$

our sol'n for x!

So our $\vec{E}$ field polarizes atoms (of course), + $\vec{p} = \dfrac{q^2 E_0/m}{\omega_0^2-\omega^2-i\gamma\omega}\, e^{-i\omega t}$

Note that $\vec{p} \propto \vec{E}$, but the proportionality is complex, $\vec{p}$ is out of phase!

Indeed, with the sign in the denominator, $\vec{p}$ "lags" $\vec{E}$ a bit.

This $\vec{p}$ adds up to a BULK volume Polarization $\vec{P} = N\,\vec{p}$

\# of molecules / volume

If each molecule has "f_j" electrons, + each electron has its own resonant frequency "ω_j" + damping "γ_j", then we really need to sum

$$\vec{P} = \frac{N q^2}{m} \sum_j \frac{f_j}{(\omega_j^2-\omega^2-i\gamma_j\omega)} \tilde{E}$$

Note: I'm assuming a dilute gas, because here only $\vec{E}_{ext}$ is polarizing. In dense materials, $\vec{P}$ creates its own $\vec{E}$ field which also polarizes. This is a Ch. 4 story, let's not fuss about it. I'm just after qualitative outcomes).

In general, $\underbrace{\vec{D} = \varepsilon\vec{E}}_{\text{linear}} = \underbrace{\varepsilon_0\vec{E} + \vec{P}}_{\text{always}} = \varepsilon_0\vec{E}\left(1 + \underbrace{\frac{Nq^2}{m}\sum_j \frac{f_j}{\omega_j^2-\omega^2-i\gamma_j\omega}}_{\text{our model}}\right)$

Nice. We have a model formula for ε, assume "dilute springlike" atoms

ε is complex!

ε depends on frequency, it's dispersive!

If you go back to our former wave eq'n in a linear dielectric,

$\nabla^2 \vec{E} = \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}$. [This is still true, all our previous work still holds, but $\varepsilon = \tilde{\varepsilon}(\omega)$ now!

So our old sol'n still holds:

$\vec{E} = \vec{\tilde{E}}_0 \, e^{i(\tilde{K} z - \omega t)}$ but, with $\frac{\tilde{K}}{\omega} = \sqrt{\tilde{\varepsilon}\mu_0}$ now

So our $\tilde{K} = \sqrt{\tilde{\varepsilon}(\omega)\cdot\omega^2\mu_0}$ is complex (which we know means losses) + depends nonlinearly on ω (dispersive)

But we learned in the last section (on conductors) how to handle complex $\tilde{K}$

$\tilde{K} = K_R + iK_{Im}$, so $\vec{\tilde{E}} = \vec{\tilde{E}}_0 \, e^{-K_{Im} z} \, e^{i(K_R - \omega t)}$

The damping arises physically from the friction (damping!) in our model,

K_R & K_{Im} are not what we had for conductors, this is totally different

they are determined from model parameters $(m_e, q, \gamma, \omega_0^2, \ldots)$

We know $\tilde{\varepsilon} = \varepsilon_0 \Big(1 + \frac{Nq^2}{\varepsilon_0 m} \underbrace{\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}}\Big)$

But my "dilute" approximation says this is small!

So $\frac{\tilde{K}}{\omega} = \sqrt{\mu_0\tilde{\varepsilon}} = \sqrt{\mu_0\varepsilon_0}\sqrt{1 + \text{something small}} \approx \sqrt{\mu_0\varepsilon_0}\left(1 + \frac{1}{2}(\text{small})\right)$

↳ use the binomial expansion!

Our model is saying

$$\frac{\tilde{K}}{\omega} = \frac{1}{c}\left(1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right), \text{(with } \tilde{K} = K_R + iK_{IM}\text{)}$$

Note that the real part of $\frac{1}{a-bi} = Re\left(\frac{1}{a-bi}\,\frac{a+bi}{a+bi}\right) = \frac{a}{a^2+b^2}$

Similarly, $IM\left(\frac{1}{a-bi}\right) = \frac{b}{a^2+b^2}$

So we can read off

$$K_R = \frac{\omega}{c}\left(1 + \frac{Nq^2}{2m\varepsilon_0}\sum_j f_j\,(\omega_j^2 - \omega^2)\Big/\left[(\omega_j^2-\omega^2)^2 + (\gamma_j\omega)^2\right]\right)$$

$$K_{IM} = \frac{\omega}{c}\,\frac{Nq^2}{2m\varepsilon_0}\sum_j f_j\,\gamma_j\,\omega\Big/\left[(\omega_j^2-\omega^2)^2 + (\gamma_j\omega)^2\right]$$

Comment: Since Intensity $\propto |E|^2$, the dying off part gives $e^{-2K_{IM}z}$

so people define $\alpha = 2K_{IM}$ = "absorption coeff". Our dielectric is absorbing energy - in a frequency dependent way! We will get "absorption lines" at certain ω's!

Indeed, $\alpha = \frac{Nq^2}{m\varepsilon_0 c}\,\omega^2 \underbrace{\sum_j f_j\,\gamma_j\Big/\left[(\omega_j^2-\omega^2)^2+(\gamma_j\omega)^2\right]}$

This is small, except that it has resonance

behaviour for $\omega = \omega_j$. So, when your E field has $\omega \simeq \omega_j$, near one of the natural resonances, this is where you get strong absorption, (+ spectroscopic absorption bands!)

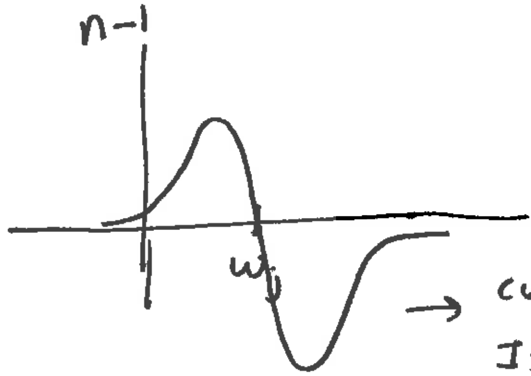
Meanwhile, the traveling wave $\sim e^{i(K_R - \omega t)}$, so

$$V_{phase} = \frac{\omega}{K_R}, \text{ + we generally define } n = \frac{c}{V_{phase}} = \frac{c K_R}{\omega}$$

In our model, $n = 1 + \underbrace{\frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}}$

Small, so $n \approx 1$.

But, positive when ω is just below a resonant ω_j



→ curious. $n < 1$ just past a resonance.
Is this bad? $n < 1$ means $V > c$!

It's OK: 1) There are still lots of other terms in that sum, they tend to form a "background" that lifts $n > 1$. So, it might dip at this frequency, but not really go < 1.

2) $V_p = c/n$ can be $> c$, as long as $V_g = \frac{d\omega}{dK} < c$.

This is a Griffiths problem, to check that this is OK in our model!

3) Note that far from all resonances, γ is typically irrelevant, +

$$n - 1 \approx \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2)}$$

For light below most "natural frequencies" (which are often UV)

$$\frac{1}{\omega_j^2-\omega^2} = \frac{1}{\omega_j^2(1-\omega^2/\omega_j^2)} \approx \frac{1}{\omega_j^2}(1+\omega^2/\omega_j^2)$$ ← Binomial Expansion

so $$n-1 \approx \frac{Nq^2}{2m\varepsilon_0}\left(\sum_j \frac{f_j}{\omega_j^2} + \omega^2 \sum_j \frac{f_j}{\omega_j^4}\right) = C_1 + C_2\omega^2$$

Turns out this is a half decent model for n of dilute gases, + those C's can be estimated (+ are of the right order of mag) experimentally)

→ This model, with $\omega \ll$ (most ω_j's) gives a rising $n(\omega)$, so you predict red light has lower n than violet, so it refracts less this is consistent with rainbows + prisms, ('tho those are not dilute!)

→ Far from resonances, $C_2 \ll C_1$, + $n \approx 1$ (gases are mostly colorless, transparent, linear media)

→ Near resonances, α gets big, you get spectral lines, + "anomalous dispersion" where n briefly drops with frequency