Electromagnetic Waves in Conductors

Exercise 1: What is $\sqrt{i}$?
Help students show $\sqrt{i} = \sqrt{e^{i\pi/2}} = e^{i\pi/4} = (1+i)/\sqrt{2}$.

Exercise 2: What about $\sqrt{a+ib}$?
Help students walk through $z^2 = a + ib = re^{-\theta}$ with $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$, so $z = \sqrt{r}e^{i\theta/2} = \sqrt{r}\,(\cos(\theta/2) + i\sin(\theta/2))$. Use $\cos(\theta/2) = \sqrt{\frac{1+\cos(\theta)}{2}}$ and $\sin(\theta/2) = \sqrt{\frac{1-\cos(\theta)}{2}}$, and $\cos(\theta) = a/r$ to give

$$\sqrt{a+ib} = \sqrt{\frac{a}{2}}\left(\sqrt{\sqrt{1+\left(\frac{b}{a}\right)^2}+1} \quad + \quad i\sqrt{\sqrt{1+\left(\frac{b}{a}\right)^2}-1}\right).$$

Exercise 3: Have students write down Maxwell's equations in media without referring to the text or notes.

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{1a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1c}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_f, \tag{1d}$$

.

Exercise 4: Have students rewrite Maxwell's equations assuming linear media and Ohm's law, where ϵ, μ, and σ are constant in space and time.

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\epsilon} \tag{2a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{2c}$$

$$\nabla \times \mathbf{B} - \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} = \mu\sigma\mathbf{E}, \tag{2d}$$

.

Exercise 5: Have students do the charge conservation in metals: dynamics exercise.

$$\nabla \cdot \mathbf{J}_f = -\frac{\partial \rho_f}{\partial t},$$
$$\nabla \cdot \mathbf{J}_f = \nabla \cdot (\sigma \ \mathbf{E}) = \sigma \nabla \cdot \ \mathbf{E} = \frac{\sigma \rho_f}{\epsilon} = -\frac{\partial \rho_f}{\partial t},$$
$$\frac{\partial \rho_f}{\partial t} + \frac{\sigma}{\epsilon}\rho_f = 0,$$
$$\rho_f(t) = \rho_f(0) \exp\left(-\frac{\sigma t}{\epsilon}\right) = \rho_f(0) \exp\left(-\frac{t}{\tau}\right),$$

where the quantity $\tau = \epsilon/\sigma$ has units of time. Using values for metals $\sigma \approx 10^7\,(\Omega\mathrm{m})^{-1}$ and $\epsilon \approx 10^{-11}\,\mathrm{C}^2/\mathrm{Nm}^2$, we get $\tau \approx 10^{-18}\,\mathrm{s}$. Therefore the free charge is very quickly eliminated from the inside of the conductor, leaving charge on surface. (Don't the excess free charges have to move all the way to the surface? Shouldn't the size of conductor enter? Walk through this. Note, this time scale is a little extreme. The physics of electron motion sets this time scale as the scattering time $\tau_s \approx 10^{-16}\,\mathrm{s}$. The paper mentioned in the footnote in text was written by Neil Ashby.)

Maxwell's equations become

$$\nabla \cdot \mathbf{E} = 0 \tag{3a}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{3b}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{3c}$$
$$\nabla \times \mathbf{B} - \mu\epsilon\frac{\partial \mathbf{E}}{\partial t} = \mu\sigma\mathbf{E}, \tag{3d}$$

Have you seen something like these before? Yes, this is just wave equation in a medium with $v = 1/\sqrt{\mu\epsilon}$, but with a new term $\mu\sigma\mathbf{E}$. Do these admit plane wave solutions?

Exercise 6: Try the plane waves:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad , \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

Have students rewrite Maxwell's equations in terms of $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{k}$, and ω.

$$i\mathbf{k} \cdot \mathbf{E}_0 = 0 \tag{4a}$$
$$i\mathbf{k} \cdot \mathbf{B}_0 = 0, \tag{4b}$$
$$i\mathbf{k} \times \mathbf{E}_0 - i\omega\mathbf{B}_0 = 0, \tag{4c}$$
$$i\mathbf{k} \times \mathbf{B}_0 + i\mu\epsilon\omega\mathbf{E}_0 = \mu\sigma\mathbf{E}, \tag{4d}$$

What can we infer about $\mathbf{E}_0$, $\mathbf{B}_0$, and $\mathbf{k}$?
From (4a) and (4b) we see that $\mathbf{k}$ is perpendicular to both $\mathbf{E}_0$ and $\mathbf{B}_0$, i.e. transverse waves.

Exercise 7: Help students solve (4c) and (4d).
Starting with the easy one (equation (4c)) , we find $\mathbf{B}_0 = \frac{1}{\omega}\mathbf{k}\times\mathbf{E}_0$. This implies that $\mathbf{B}_0$ is perpendicular to $\mathbf{E}_0$, just like in vacuum. Now substitute into (4d) to get

$$\frac{i}{\omega}\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0) + i\mu\epsilon\omega\mathbf{E}_0 = \mu\sigma\mathbf{E}_0.$$

Using $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, we get

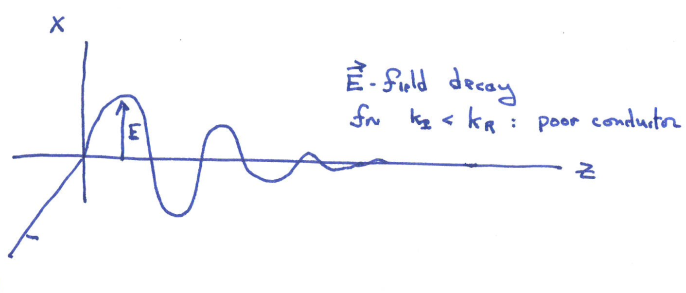
$$\frac{-ik^2}{\omega}\mathbf{E}_0 + i\mu\epsilon\omega\mathbf{E}_0 = \mu\sigma\mathbf{E}_0,$$

which gives

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega. \tag{5}$$

Exercise 8: What are the consequences?

1. The wavevector k is complex, i.e. $k = k_I + ik_R$.
2. This gives exponentially decaying solutions for $\mathbf{E}$ and $\mathbf{B}$ with exponentials of the form $\exp(-k_I z)\exp(i(k_R z - \omega t))$. The length scale of the decay is called the skin depth: $d = 1/k_I$, and the phase velocity of the decaying waves becomes ω/k_R.



Since $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{k}$ are mutually perpendicular, lets choose $\mathbf{k}$ along $\hat{z}$, $\mathbf{E}_0$ along $\hat{x}$ and $\mathbf{B}_0$ along $\hat{y}$. This gives

$$\mathbf{E} = E_0\hat{x}\exp(-k_I z)\exp(i(k_R z - \omega t)), \tag{6a}$$

$$\mathbf{B} = \frac{k_R + ik_I}{\omega}E_0\hat{y}\exp(-k_I z)\exp(i(k_R z - \omega t)). \tag{6b}$$

From this, we can see that $\mathbf{B}$ is out of phase with $\mathbf{E}$. We can solve equation (5) for k_R and k_I in general, but lets stick to the case of good conductors at visible frequencies and below. Rewriting equation (5) a little, $k^2 = \mu\epsilon\omega^2 + i\mu\epsilon\omega\frac{\sigma}{\epsilon}$, we see that the second term on the right hand side dominates for visible frequencies and below since $\sigma/\epsilon \approx 10^{18}\,\mathrm{s}^{-1}$, so the dispersion relation becomes $k^2 = i\mu\omega\sigma$. Have students solve this using Exercise 1. The solution is $k = \sqrt{\mu\omega\sigma}(1+i)/\sqrt{2}$ so the real and imaginary parts are equal:

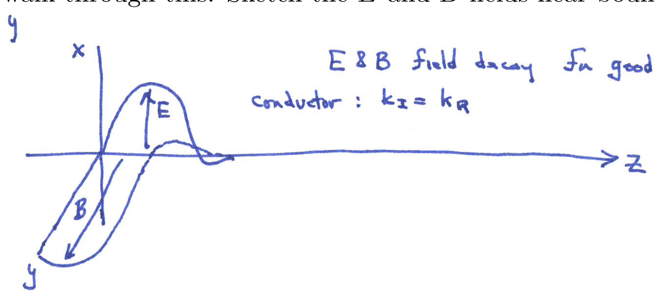
$$k_R = k_I = \sqrt{\frac{\mu\omega\sigma}{2}}. \tag{7}$$

Putting in $\sigma \approx 10^7\,(\Omega\mathrm{m})^{-1}$, $\mu \approx \mu_0 = 4\pi 10^{-7}\,\mathrm{N/A^2}$, and typical frequencies for visible light ($\omega \approx 10^{15}\,\mathrm{s}^{-1}$), microwaves ($\omega \approx 10^{10}\,\mathrm{s}^{-1}$), and AC electrical systems ($\omega = 2\pi(60)\,\mathrm{s}^{-1}$), we get skin depths of $d_{visible} \approx 10\,\mathrm{nm}$, $d_{microwave} \approx 4\,\mu\mathrm{m}$, and $d_{AC} \approx 2\,\mathrm{cm}$. Discuss applications: eclipse glasses, mirrors, half-silvered mirrors, microwave cables, power line cables.

Exercise 9: We can see that **B** is out of phase with **E**, but by how much? For good conductors

$$B_0 = \frac{k_R}{\omega}(1+i)E_0 = \sqrt{\mu\omega\sigma}E_0 e^{i\pi/4},$$

so the phase difference is $\pi/4$, but does the B-field lead or lag the E-field? Help students walk through this. Sketch the E and B fields near boundary.



How does the magnitude of the B-field compare to the E-field?

$$\left|\frac{cB_0}{E_0}\right| \approx \frac{ck_r}{\omega} \approx \sqrt{\frac{c^2\mu\epsilon\sigma}{\epsilon\omega}} \approx \sqrt{\frac{\sigma}{\epsilon\omega}} \gg 1.$$

Therefore, the ratio of B to E is much larger than in the vacuum, and nearly all of the energy density is in the B-field.