Computing induced E-fields

New equations for $\vec{E}$:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

"Pure Faraday field" $\rightarrow \rho = 0 \rightarrow \vec{\nabla}\cdot\vec{E} = 0 \;;\; \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

$\Updownarrow$ Mathematically identical

$\vec{\nabla}\cdot\vec{B} = 0 \;,\; \vec{\nabla}\times\vec{B} = \mu_0\vec{J}.$

- Solving for $\vec{E}$ given $\dfrac{\partial\vec{B}}{\partial t}$ $\Leftrightarrow$ Solving for $\vec{B}$ given $\vec{J}$ in magnetostatics.

Here if we are given $\dfrac{\partial\vec{B}}{\partial t}$, no add^{l} assumptions needed. $\dfrac{\partial\vec{B}}{\partial t}$ can be time-dependent.

Here must assume magnetostatics, since Ampere's Law will get modified beyond statics.

So to find $\vec{E}$, what to do?

(1) $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} \Leftrightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

w/ enough symmetry, proceed as with Ampere's Law.

2) Also analog of Biot - Savart.

Magnetostatics : $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{A} = 0$, $-\nabla^2 \vec{A} = \mu_0 \vec{J}$.

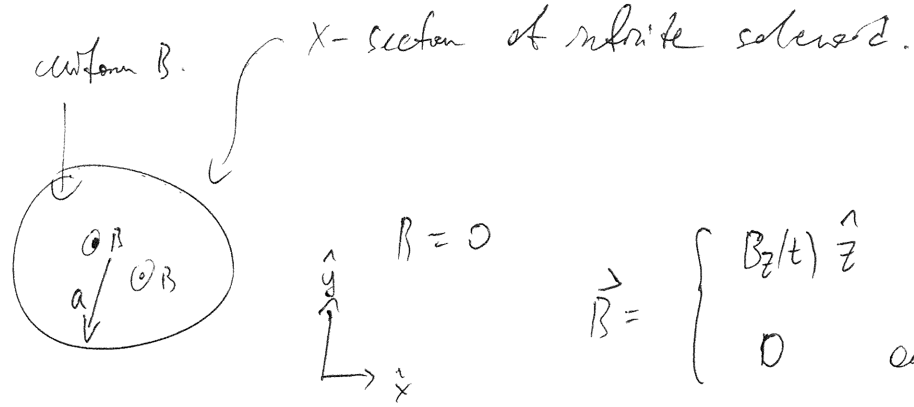
$$\Rightarrow \vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')\,d\tau'}{\mathcal{r}} \Rightarrow \vec{\nabla} \times \vec{A} = \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathcal{r}}}{\mathcal{r}^2}\, d\tau'.$$

(Can show using $\vec{\nabla} \times \left(\frac{1}{\mathcal{r}} \underbrace{\vec{J}(\vec{r}\,')}_{\text{indep. of } \vec{r}} \right) = \frac{\vec{J}(\vec{r}\,') \times \hat{\mathcal{r}}}{\mathcal{r}^2}$.)

Faraday E-field : $\vec{E} = \vec{\nabla} \times \vec{A}$, $\vec{\nabla} \cdot \vec{A} = 0$, $-\nabla^2 \vec{A} = -\frac{\partial \vec{B}}{\partial t}$.

$$\Rightarrow \vec{A} = -\frac{1}{4\pi} \int \frac{\partial \vec{B}/\partial t (\vec{r}\,')\, d\tau'}{\mathcal{r}} \Rightarrow \vec{E} = \vec{\nabla} \times \vec{A} = -\frac{1}{4\pi} \int \frac{\frac{\partial \vec{B}}{\partial t} \times \hat{\mathcal{r}}}{\mathcal{r}^2}\, d\tau'.$$

Example:



$$\vec{B} = \begin{cases} B_z(t)\,\hat{z} & \text{inside} \\ 0 & \text{outside.} \end{cases}$$

Find $\vec{E}(\vec{r}, t)$.

- Analogous to finding $\vec{B}$ for circular wire carrying uniform $\vec{J}$.
- Cylindrical coordinates:

$$\vec{E} = E_s \hat{s} + E_\phi \hat{\phi} + E_z \hat{z}.$$

Claim $E_s = E_z = 0$. Let's show this with a careful symmetry analysis.

- Rotational (axial) symmetry ($\phi \to \phi + \text{const}$) $\Rightarrow$ $E_\phi = E_\phi(s)$
  + translational symmetry ($z \to z + \text{const}$)

$$E_s = E_s(s)$$
$$E_z = E_z(s)$$

• Inversion symmetry : $\vec{r} \to -\vec{r}$.

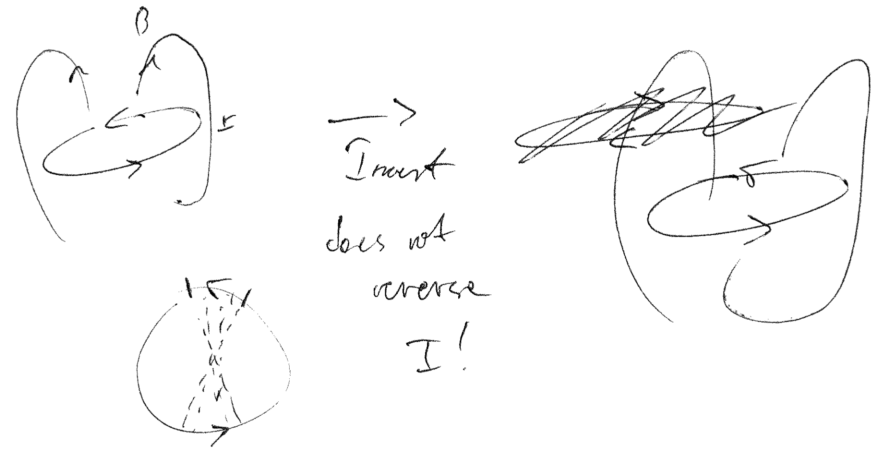
"vector" $\qquad\qquad\qquad\qquad$ "pseudovector"

$\vec{E} \to -\vec{E}$, but $\vec{B} \to \vec{B}$, so this situation is inversion symmetric.

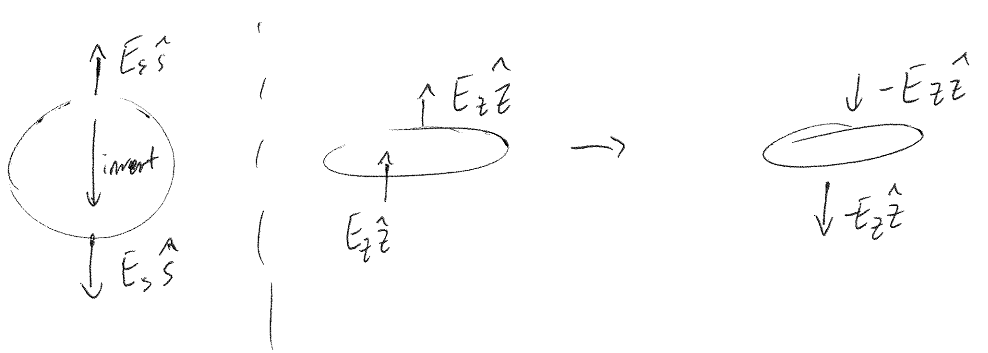
Why is $\vec{B}$ pseudovector :

(1) $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ only consistent w/ inversion symm. if $\vec{B} \to \vec{B}$.

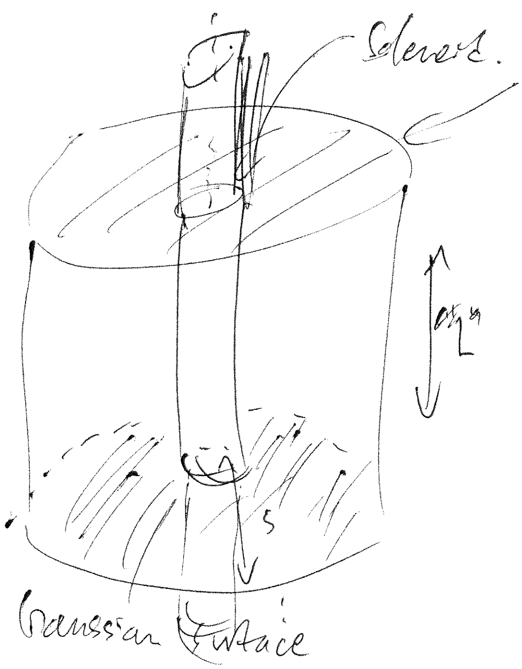
(2)



Here, $\Rightarrow$ $E_z = 0$. Implies nothing for E_s.



• Why is $E_s = 0$? Not symmetry, but Gauss' law:



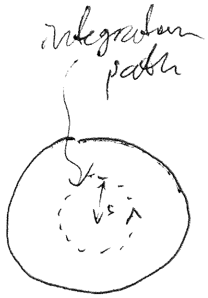
no contribution from ends.

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = 0$$

$$\parallel$$

$$2\pi s L E_s \quad \Rightarrow E_s = 0.$$

Back to finding $\vec{E} = E_\phi(s)\,\hat{\phi}$.



$$\oint \vec{E}\cdot d\vec{l} = 2\pi s E_\phi = -\frac{d\Phi}{dt}$$

$$= -\begin{cases} \pi s^2 \dfrac{dB_z}{dt} & ; \; s<a \\ \pi a^2 \dfrac{dB_z}{dt} & ; \; s>a. \end{cases}$$

$$\Rightarrow \boxed{\vec{E} = \begin{cases} -\dfrac{s}{2}\dfrac{dB_z}{dt}\hat{\phi} & ; \; s<a \\ -\dfrac{a^2}{2s}\dfrac{dB_z}{dt}\hat{\phi} & ; \; s>a \end{cases}}$$