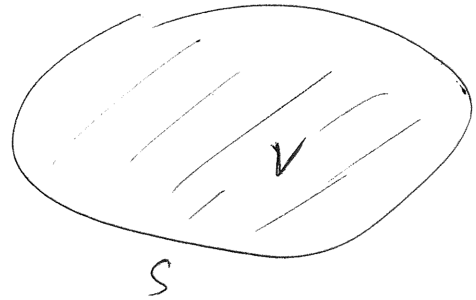
A Uniqueness Theorem

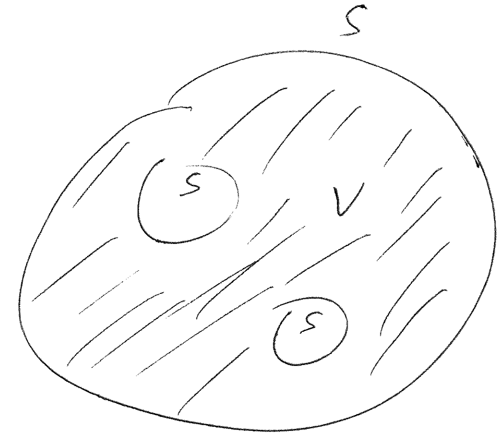
time-independent.

- Consider a ~~cg~~ region V with some charge density $\rho(\vec{r})$ inside, bounded by surface S. ~~Everything is static.~~

e.g.



or



, etc.

Inside this region, suppose $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ $\Rightarrow$ $\vec{E} = -\vec{\nabla}V$, $-\nabla^2 V = \frac{\rho}{\varepsilon_0}$.

- Boundary conditions:

$S \to$ two parts.

S_V, on which V is specified.

$S_{\vec{E}}$, on which $\frac{\partial V}{\partial n}$ is specified.

normal derivative of V at surface

$\vec{E}\cdot\hat{n} = -\frac{\partial V}{\partial n}$

~~Elalon~~

Claim: There is a unique $\vec{E}$-field in V satisfying the electrostatic equations and the specified boundary conditions.

Proof: Suppose we have 2 solutions V_1 and V_2.

Then $V_3 = V_1 - V_2$ $\left(\vec{E}_3 = \vec{E}_1 - \vec{E}_2 \right)$ is a solution to Laplace's equation in V: $-\nabla^2 V_3 = 0$;

with boundary conditions
$$V_3 = 0 \text{ on } S_V$$
$$\vec{E}_3 \cdot \hat{n} = 0 \text{ on } S_{\vec{E}}.$$

Trick: $\vec{\nabla} \cdot (V_3 \vec{E}_3) = V_3 \cancelto{0}{(\vec{\nabla} \cdot \vec{E}_3)} + \vec{E}_3 \cdot (\vec{\nabla} V_3) = -E_3^2.$

$$\Rightarrow \quad \int_V \vec{\nabla} \cdot (V_3 \vec{E}_3)\, d\tau = -\int_V E_3^2\, d\tau$$

$$\| $$

$$\oint_S V_3 \vec{E}_3 \cdot d\vec{a} = 0,$$ since either $V_3 = 0$ or $\vec{E}_3 \cdot d\vec{a} = 0$ anywhere on S.

$$\Rightarrow \vec{E}_3 = 0 \quad \Rightarrow \quad \vec{E}_1 = \vec{E}_2. \quad \boxtimes$$