1.2. Gauss's Law (and Faraday's Law)

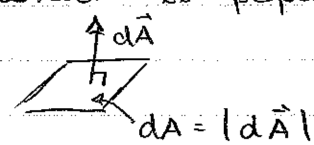
Coulomb's Law and the related equation(s) for the electric field for a charge distribution are the basics of electrostatics. Unfortunately the integrals needed to compute the electric field are often difficult to calculate. We will therefore develop some methods to avoid the integrals (in certain cases). The first of it is Gauss's law. It can be written in two forms:

Integral form: $$\oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} \qquad \left(k = \frac{1}{4\pi\varepsilon_0}\right)$$

Differential form: $$\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

Let us clarify the symbols:

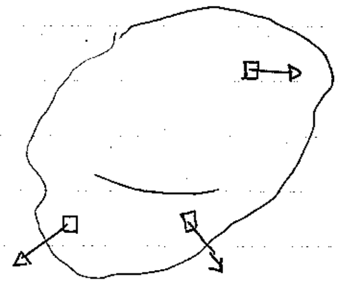
$d\vec{A}$ is an area vector, which is perpendicular to the area element dA



$\vec{E} \cdot d\vec{A} = E\,dA \cos\Theta$, where Θ is the angle between $\vec{E}$ and $d\vec{A}$, is the electric flux through the area element dA

The electric flux is a measure of the electric field lines passing through the (infinitesimal) area element $d\vec{A}$. Since the density of the field lines is proportional to the field strength at a certain point $\vec{r}$, the electric flux through a closed surface S is a measure of the total charge $Q_{enclosed}$ enclosed. This is Gauss's law in integral form.

Before we continue in proving Gauss's law we need to introduce as a convention that the area vector for a closed surface is directed outward and

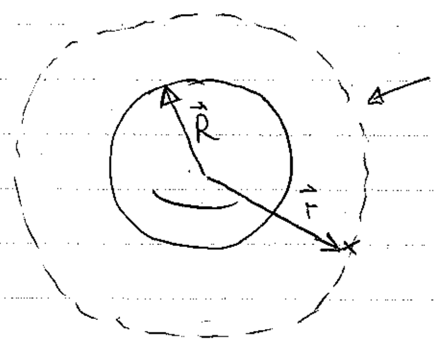


$\vec{E} \cdot d\vec{A} > 0$ if $\vec{E}$ is directed outward as well

$\vec{E} \cdot d\vec{A} < 0$ if $\vec{E}$ is directed inward

Let us also first see an application of Gauss's law

Example: Spherical shell, uniformly charged surface density σ
Determine $\vec{E}$ outside of shell



imaginary surface (sphere) S often also called Gaussian sphere

Symmetry tells us $\vec{E}$ points radially outward: $\vec{E}(\vec{r}) = E(r)\,\hat{r}$

Gauss's law: $$\oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{4\pi R^2 \sigma}{\varepsilon_0}$$

$$= \oint_S E(r) \underbrace{\hat{r} \cdot d\vec{A}}_{= 1\, dA}$$

$$= E(r) \oint 1\, dA = E(r) \overbrace{4\pi r^2}^{\text{Surface of imaginary spherical shell of radius } r}$$

$$\Rightarrow E(r) \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \qquad \Rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}\,\hat{r}$$

Now we proove Gauss's law:

We know: $\vec{E}(\vec{r}) = k \int_v \frac{\hat{R}}{R^2} \rho(\vec{r}\,') \, dV'$

We want for Gauss's Law (in differential form): $\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r})$

Therefore: $\vec{\nabla}_r \cdot \vec{E}(\vec{r}) = k \underbrace{\vec{\nabla}_{\vec{r}}}_{} \cdot \int_v \frac{\hat{R}}{R^2} \rho(\vec{r}\,') \, dV'$

can be moved inside integral (over dV')

$$= k \int_v \underbrace{\left(\vec{\nabla}_{\vec{r}} \cdot \frac{\hat{R}}{R^2} \right)}_{= \vec{\nabla}_{\vec{R}} \cdot \frac{\hat{R}}{R^2}} \rho(\vec{r}\,') \, dV'$$

(What is this?)

<u>Math excursus</u>

product rule

$$\vec{\nabla}_R \cdot \frac{\hat{R}}{R^2} = \vec{\nabla}_R \cdot \frac{\vec{R}}{R^3} = \left(\vec{\nabla}_{\vec{R}} \frac{1}{R^3} \right) \cdot \vec{R} + \frac{1}{R^3} \left(\vec{\nabla}_{\vec{R}} \cdot \vec{R} \right)$$

$$= -3 \frac{\hat{R}}{R^4} \vec{R} + \frac{1}{R^3} 3 = 0$$

Well, this calculation is correct except for $R = 0$, where the above is not well defined.

In order to analyze the situation at $R=0$ we use the divergence theorem, which states

$$\int_V \vec{\nabla} \cdot \vec{v} \, dV = \int_S \vec{v} \cdot d\vec{A}$$

S: surface which bounds volume V

Here $\vec{v} = \frac{\hat{R}}{R^2}$ and V needs to include $R=0$ and be a sphere

$$\int_V \vec{\nabla} \cdot \frac{\hat{R}}{R^2} \, dV = \oint_S \frac{\hat{R}}{R^2} \underbrace{d\vec{A}}_{R^2 \sin\Theta \, d\Theta \, d\Phi \, \hat{R}}$$

$$= \frac{R^2}{R^2} \oint_S 1 \sin\Theta \, d\Theta \, d\Phi = 4\pi$$

Thus, the volume integral over our function $\vec{\nabla} \cdot \frac{\hat{R}}{R^2}$ which is zero everywhere except at $R=0$ (single point) gives a finite non-zero value. Strange !?

Such a "function" is known to involve the Dirac δ-function:

In $\underline{1D}$ $\quad \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$

$$\text{and} \int_{-\infty}^{\infty} \delta(x) \, dx = 1$$

This is not a real function (value at $x=0$ is not finite), it is called generalized function or distribution.

δ-functions can be considered as limit of a sequence of function which increasingly peak at $x=0$ and the the integral from $-\infty$ to ∞ is always 1

Examples: $$\delta(x) = \lim_{\varepsilon \to 0} \frac{\varepsilon/\pi}{x^2+\varepsilon^2}$$

or $$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{\pi}\,\varepsilon} \exp\left(-\frac{x^2}{\varepsilon^2}\right)$$

or $$\delta(x) = \lim_{\varepsilon \to 0} \frac{\sin(x/\varepsilon)}{\pi x}$$ and many more.

These sequences are also called representations of δ-function. There are also integral representation of δ-function, e.g.

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx)\, dx$$

In combination with other functions we get

$$f(x)\,\delta(x) = f(x=0)\,\delta(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(x=0) \underbrace{\int_{-\infty}^{\infty} \delta(x)\,dx}_{=1} = f(x=0)$$

The position of the "infinite peak" can be shifted

$$\delta(x=a) = \begin{cases} 0 & \text{for } x \neq a \\ \infty & \text{for } x = a \end{cases}$$

with $\displaystyle\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$

and $\displaystyle\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(x=a)$

The δ-function in 3D is given as

$$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

or $\delta^3(\vec{r}-\vec{a}) = \delta(x-a_x)\,\delta(y-a_y)\,\delta(z-a_z)$

with $\displaystyle\int_V \delta^3(\vec{r})\,dV = \int_V \delta^3(\vec{r}-\vec{a})\, dV = 1$

↑ enclosing point of "infinite peak"

$$\Rightarrow \int_V f(\vec{r})\, \delta^3(\vec{r}-\vec{a})\, dV = f(\vec{a})$$

Thinking back of our function $\vec{\nabla} \cdot \frac{\hat{R}}{R^2}$ the δ-function is part of the solution. With

$$\vec{\nabla} \cdot \frac{\hat{R}}{R^2} = 4\pi \, \delta^3(\vec{R}) \quad \text{we have}$$

$$\int_{\mathcal{V}} \vec{\nabla} \cdot \frac{\hat{R}}{R^2} \, dV = \int_{\mathcal{V}} 4\pi \, \delta^3(\vec{R}) \, dV = 4\pi$$

while $4\pi \, \delta^3(\vec{R})$ is zero everywhere except at the origin.

After this math excursion, we can continue with our proof of Gauss's law.

$$\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r}) = k \int_{\mathcal{V}} \underbrace{\left(\vec{\nabla}_{\vec{R}} \, \frac{\hat{R}}{R^2} \right)}_{= 4\pi \, \delta(\vec{R})} \rho(\vec{r}\,') \, dV'$$

$$= \frac{4\pi}{4\pi \varepsilon_0} \int_{\mathcal{V}} \delta(\vec{r} - \vec{r}\,') \, \rho(\vec{r}\,') \, dV'$$

$$= \frac{\rho(\vec{r})}{\varepsilon_0}$$: Gauss's law in differential form.

To get Gauss's law in integral form we use the divergence theorem.

$$\vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\Rightarrow \underbrace{\int_{\mathcal{V}} \vec{\nabla}_{\vec{r}} \cdot \vec{E}(\vec{r})\, dV}_{= \oint_S \vec{E}(\vec{r}) \cdot d\vec{A}} = \underbrace{\int_{\mathcal{V}} \frac{\rho(\vec{r})}{\varepsilon_0}\, dV}_{= \frac{1}{\varepsilon_0} Q_{\text{enclosed in } \mathcal{V}}}$$

divergence theorem

$$\Rightarrow \oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Gauss's law in integral form.

Before we proceed with applications of Gauss's law let us see some applications of the δ-function in writing volume charge densities $\rho(\vec{r})$ for certain charge distributions.

point charge Q at $P = (2, -1, 3)$

$$\rho(\vec{r}) = Q\,\delta(x-2)\,\delta(y+1)\,\delta(z-3)$$

$$\text{since} \int_{\mathbb{R}^3} \rho(\vec{r})\,d\vec{r} = \int_{\mathbb{R}} Q\,\delta(x-2)\,\delta(y+1)\,\delta(z-3)\,dV = Q$$

Do you see that each of the δ-functions has units $\left[\frac{1}{m}\right]$

line charge density $\lambda(\vec{r})$ parallel to x-axis through $P(x,-1,3)$

$$\rho(\vec{r}) = \lambda(\vec{r})\,\delta(y+1)\,\delta(z-3)$$

surface charge density $\sigma(\vec{r})$ parallel to x-y plane through $P(x,y,3)$

$$\rho(\vec{r}) = \sigma(\vec{r})\,\delta(z-3)$$

Figure the integrals and units out by yourself.

Gauss's Law is true always (even in case of moving charges) but it is unfortunately useful only sometimes, since the electric field appears as an integrand (or after a differential operator) and we need to "pull" it out".

Think again about our first application (spherical shell uniformly charged). We use the symmetry of the problem to see

$$\vec{E}(\vec{r}) = E(r)\,\hat{r}$$

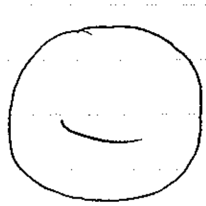
and then constructed an (imaginary) "Gaussian" surface as a sphere such that $\vec{E} \parallel d\vec{A}$ everywhere

$$\Rightarrow \oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = E(r) \oint_S \underbrace{\hat{r} \cdot d\vec{A}}_{= 1\, dA} = E(r)\, 4\pi r^2$$

surface of sphere ($4\pi r^2$)

Essentially Gauss's Law can be used in cases of high symmetry and (at least partially) uniform charge distributions.

Spherical



$\vec{E}(\vec{r}) = E(r)\,\hat{r}$

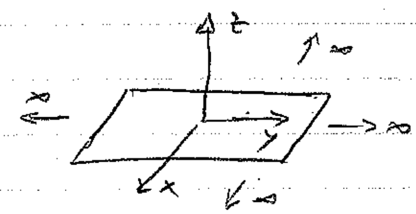
charge distribution needs to be uniform in θ, ϕ not necessarily in r

Cylindrical

$\vec{E}(\vec{r}) = E(s)\,\hat{s}$

uniform in θ and z not necessarily in s

Planar



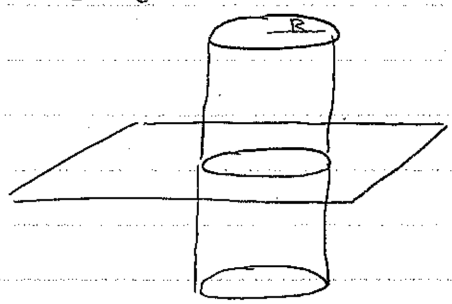
$$\vec{E}(\vec{r}) = \begin{cases} E(z)\,\hat{z}, & z>0 \\ -E(z)\,\hat{z}, & z<0 \end{cases}$$

uniform in x, y not necessarily in z

Let's consider one more example, namely a sheet with uniform surface density σ in x, y - plane

As mentioned above,
$$\vec{E}(\vec{r}) = \begin{cases} E(z)\ \hat{z} & \text{for } z > 0 \\ -E(z)\ \hat{z} & \text{for } z < 0 \end{cases}$$

Choose e.g. "Gaussian" surface with areas parallel and perpendicular to $\hat{z}$



Gauss's Law:
$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{\sigma \pi R^2}{\varepsilon_0}.$$

$$= \int_{\text{Top area}} E(z)\ \underbrace{\hat{z}\cdot d\vec{A}}_{=\pi R^2} + \int_{\text{Bottom}} -E(z)\ \underbrace{\hat{z}\cdot d\vec{A}}_{=-\pi R^2} + \int_{\text{Sides}} E(z)\ \underbrace{\hat{z}\cdot d\vec{A}}_{=0}$$

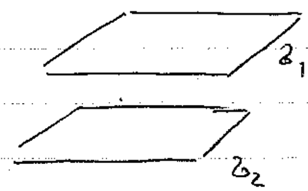
$$= \Big(E(z) + E(z)\Big)\ \pi R^2 = 2\,E(z)\ \pi R^2$$

$$\Rightarrow E(z) = \frac{\sigma \pi R^2}{2\varepsilon_0 \pi R^2} \quad \Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\ \hat{z} & \text{for } z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\ \hat{z} & \text{for } z < 0 \end{cases}$$

A few other problems can be solved by superposition:

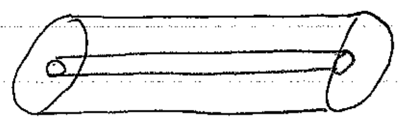
Examples

"Condensator"



$$\vec{E}(\vec{r}) = \vec{E}_{\text{due to } \sigma_1}(\vec{r}) + \vec{E}_{\text{due to } \sigma_2}(\vec{r})$$

"Coaxial cable"



Gauss's law contains the divergence of $\vec{E}$, we will now derive another law which contains the curl of $\vec{E}$ namely it says

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

We know: $\vec{E}(\vec{r}) = k \int_V \frac{\hat{\mathcal{R}}}{\mathcal{R}^2} \rho(\vec{r}')\, dV'$

$$\Rightarrow \vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r}) = k \int_V \left(\vec{\nabla}_{\vec{r}} \times \frac{\hat{\mathcal{R}}}{\mathcal{R}^2} \right) \rho(\vec{r}')\, dV'$$

(same arguments as in derivation of Gauss's Law)

At this point we need that

$$\vec{\nabla}_{\vec{r}} \times \left(\vec{\nabla}_{\vec{r}}\, f(\vec{r}) \right) = 0 \quad \text{for any scalar function } f(\vec{r})$$

<u>Proof</u>: (for z-component)

$$\left(\vec{\nabla} \times A\right)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad \text{and} \quad \vec{A} = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\Rightarrow \left(\vec{\nabla} \times \left(\vec{\nabla} f \right) \right)_z = \frac{\partial f}{\partial x \partial y} - \frac{\partial f}{\partial y \partial x} = 0$$

Since $\vec{\nabla}_{\vec{r}} \frac{1}{R} = \vec{\nabla}_{\vec{R}} \frac{1}{R} = -\frac{1}{R^2} \hat{R}$

$$\Rightarrow \vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r}) = k \int_V \underbrace{\left(\vec{\nabla}_{\vec{r}} \times \frac{\hat{R}}{R^2} \right)}_{= -\vec{\nabla}_{\vec{r}} \times \left(\vec{\nabla} \frac{1}{R} \right) = 0} \rho(\vec{r}\,')\, dV'$$

use math theorem above

$$= 0$$

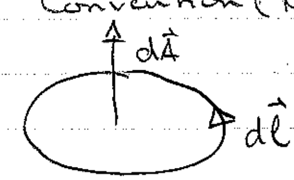
This is the differential form of the law. As for Gauss's law there exist also an integral form, which we get by using the curl theorem (or Stoke's theorem)

$$\int_S \left(\vec{\nabla} \times \vec{v} \right) d\vec{A} = \oint_P \vec{v} \cdot d\vec{l}$$

S: open surface

P: path along boundaries of S

Convention (Right hand rule):



Here: $\vec{v} = \vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r})$

$$\int_S \left(\vec{\nabla}_{\vec{r}} \times \vec{E}(\vec{r}) \right) \cdot d\vec{A} = 0$$

$$\Rightarrow \oint_P \vec{E} \cdot d\vec{l} = 0$$

Summary:

	Integral form	Differential form
Gauss's law	$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$	$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho}{\varepsilon_0}$
(Faraday's) law	$\oint_P \vec{E} \cdot d\vec{l} = 0$	$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$

These are nothing else than the first two Maxwell Eqns for the electrostatic case.

Compare: $\vec{\nabla} \cdot \vec{E}(\vec{r}) = \rho/\varepsilon_0$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -\frac{\partial \vec{B}}{\partial t}$$ (in electrostatic there is no $\vec{B}$-field)

In electrodynamics the 2nd law is also called Faraday's law.