6.4. Energy in magnetic fields

For a stationary current in a conductor (e.g., a loop of wire) we have Ohm's Law:

\[ V_{ext} = R \cdot I \]

If the magnetic flux \( \Phi \) through the surface of the wire changes (e.g., by a change of current in the wire), then according to Faraday's Law, there is an additional electromotive force induced and we get:

\[ V_{ext} + \varepsilon = R \cdot I \quad \Rightarrow \quad V_{ext} = R \cdot I + \frac{d\Phi}{dt} \]

\[ = R \cdot I + L \frac{dI}{dt} \]

Before we will analyze in the next subsection the time dependence of the current in such a circuit, we will have a quick look at a different aspect. Multiplying the equation by \( I \), we get a relation for the power in the system:

\[ V_{ext} \cdot I = R \cdot I^2 + L \cdot I \frac{dI}{dt} \]

\[ \text{external power} \quad \text{powers dissipated} \]

\[ \text{provided e.g. by in resistance of wire} \quad \text{What is this term?} \]
We can rewrite this last term as

\[
\frac{d}{dt} \left( \frac{1}{2} LI^2 \right)
\]

\[
\text{Power} = \frac{\text{Energy}}{\text{Time}}
\]

this must be an energy term.

The energy \( \frac{1}{2} LI^2 \) should be somehow related to the magnetic field and the magnetic flux, which induces the electromotive force. Let's see how this works out, in other words: Can we relate \( \frac{1}{2} LI^2 \) to \( \Phi_B \)?

\[
\frac{1}{2} LI^2 = \frac{1}{2} L \Phi_B
\]

since \( \mathcal{E} = -L \frac{dI}{dt} = -d \left( LI \right) = -d \Phi_B \)

\[
\Rightarrow \Phi_B = LI
\]

\[
= \frac{1}{2} \int \int \mathbf{A} \cdot d\mathbf{l}
\]

since \( \Phi_B = \int \mathbf{A} \cdot d\mathbf{l} \)

\[
= \frac{1}{2} \int \int \mathbf{A} \cdot (\nabla \times \mathbf{B}) d\mathbf{r}
\]

since \( \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \)

So far we achieved to rewrite the current and self-inductance using the vector potential \( \mathbf{A} \) and the magnetic field \( \mathbf{B} \) (via magnetic flux \( \Phi_B \) and current density \( \mathbf{j} \)). Now, we
need to express $\vec{A}$ with $\vec{B}$. Since we know $\vec{B} = \vec{\nabla} \times \vec{A}$, we would like to get the differential operator from $(\vec{\nabla} \times \vec{B})$ to $(\vec{\nabla} \times \vec{A})$. A product rule for differential operators helps, namely

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\Rightarrow \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

And we get

$$\frac{1}{2} I_2 = \frac{1}{2} \mu_0 \left[ \int_V \vec{B} \cdot (\vec{\nabla} \times \vec{A}) \, d^3 \vec{r} - \int_V \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \, d^3 \vec{r} \right] = \vec{B}$$

The second term in the bracket can be written as

$$\int_V \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \, d^3 \vec{r} = \oint_s (\vec{A} \times \vec{B}) \, d\vec{A}$$

Volume which contains surface bounding volume which contains current, i.e., we can consider all space containing current. For localized currents $\vec{A} \to 0$ at surface of whole space

$$\Rightarrow W_{\text{magnetic}} = \frac{1}{2} I_2 = \frac{1}{2} \mu_0 \int_V |\vec{B}|^2 \, d^3 \vec{r}$$

Thus, it is an energy which is stored in the magnetic field!