Notes on Lie groups and Lie algebras for PHYS5030

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1 Examples of Lie groups and Lie algebras

1.1 SO(N)

Now that we've discussed Lie groups and Lie algebras in general to some extent, let's look at some examples. We will start with SO(N), the group of rotation matrices in N-dimensional space. Recall that elements of SO(N) are just real $N \times N$ matrices R that satisfy $R^T R = I$, where I is the $N \times N$ identity matrix, and also $\det R = 1$. (If we drop the condition $\det R = 1$, then we have the group O(N).)

Our general discussion required us to start with a Lie group that has a faithful unitary representation, and we already have that here, because the rotation matrices R are already unitary. So for an infinitesimal rotation we can write $R = I + i\alpha_a X_a$, where the X_a are Hermitian. The rotation matrices are a subset of $N \times N$ unitary matrices, so there will be more conditions on X_a. In particular since R is real, X_a has to be purely imaginary. A purely imaginary Hermitian matrix is necessarily antisymmetric, so $X_a^T = -X_a$. In fact this is the only condition we need because

$$R^T R = (I + i\alpha_a X_a^T)(I + i\alpha_b X_b) = I + i\alpha_a (X_a^T + X_a), \tag{1.1}$$

dropping second order terms for infinitesimal α_a. As long as X_a is antisymmetric, the right-hand side is just the identity matrix I. Therefore the generators of SO(N) are just antisymmetric Hermitian matrices. We will pick X_a to be a basis for such matrices (see below). We won't try to prove it, but it's true that a general SO(N) matrix can be written in the form

$$R = \exp(i\alpha_a X_a). \tag{1.2}$$

What about the condition $\det R = 1$? There is a useful matrix identity,

$$\det e^A = e^{\operatorname{tr} A}. \tag{1.3}$$

This is easy to prove if A is Hermitian (or anti-Hermitian) – go to a basis where A is diagonal, and the identity is obviously true in this basis. Then observe that both the determinant and the trace are basis-independent, so it has to be true in any basis.

In the present case, since any anti-symmetric matrix X_a is traceless, we have $\det R = \exp(i\alpha_a \operatorname{tr} X_a) = e^0 = 1$. So any matrix of the form Eq. (1.2) is automatically in SO(N), *i.e.*

it automatically has determinant 1, even though we did not use the determinant 1 condition in figuring out the generators. Since we didn't use the determinant 1 condition, you might have thought we would get all orthogonal matrices. The reason we don't get orthogonal matrices with determinant -1 is that these are not continuously connected to the identity matrix, but any matrix of the form Eq. (1.2) is obviously continuously connected to the identity (just make α_a smaller and smaller). To recap what happened here, the groups SO(N) and O(N) have the same generators, but if we exponentiate the generators, we only get SO(N) matrices. This makes sense, because it turns out that SO(N) is precisely the subgroup of O(N) consisting of those orthogonal matrices that can be continuously deformed to the identity matrix.

To get a handle on the Lie algebra of SO(N), it's helpful to choose a particular set of X_a's. First let's look at $N = 2$. In this case, up to a proportionality constant, there is only one 2×2 Hermitian antisymmetric matrix,

$$X_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{1.4}$$

The only thing to say about the SO(2) Lie algebra is the obvious fact that $[X_1, X_1] = 0$. This just reflects the fact that rotations of two-dimensional space always commute with each other.

Let's move on to $N = 3$, which is more interesting. Then there are three generators, and we choose

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \equiv L_1 \tag{1.5}$$

$$X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \equiv L_2 \tag{1.6}$$

$$X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv L_3. \tag{1.7}$$

In Zee's book, these matrices are called J_x, J_y and J_z, respectively, and they generate rotations about the x, y and z axes. (You should check this *e.g.* by working out the matrix exponential $\exp(i\theta L_3)$, which corresponds to a rotation by θ about the z-axis. You can work this out directly from the Taylor series that defines the matrix exponential.) These matrices obey the familiar commutation relations

$$[L_a, L_b] = i\epsilon_{abc} L_c, \tag{1.8}$$

where ϵ_{abc} is the usual completely antisymmetric ϵ-symbol. So evidently for SO(3), we have found the structure constants of the Lie algebra, and $f_{abc} = \epsilon_{abc}$. It's important to note that the structure constants depended on making a particular choice of the X_a's, a basis choice if you like. We could have chosen a different set of three X_a's, as long as any Hermitian antisymmetric matrix can be written as a linear combination of them (with real coefficients), and this would be fine but the expression for the structure constants would be different in this different basis.

Just a quick word about notation here – I am trying to use X_a to refer to the generators of a Lie algebra in general, but use other symbols (at least sometimes) when referring to specific generators of some specific Lie algebra. That's why I called the generators L_1, L_2, L_3 above.

Now let's discuss the generators and the Lie algebra for general N. In N-dimensional space, the analog of rotations about the x, y and z axes in three-dimensional space are rotations in the (mn)-plane, where $m, n = 1, \ldots, N$ and we take $m < n$. In such a rotation, we rotate the m-axis into the n-axis and vice versa. So for instance, a rotation about the z-axis in three dimensional space can also be described as a rotation in the (12)-plane. This kind of description is better for $N > 3$, because for $N > 3$, we can't specify a plane by "the" axis perpendicular to it – there are many axes perpendicular to each plane. So to write down the generators of SO(N), we trade the label a for the new label (mn) ... these are just different ways of labeling the generators, so we can use whatever notation we like. Then we will have generators $X_{(mn)} \equiv L_{(mn)}$, which are given by the formula

$$(L_{(mn)})_{ij} = -i(\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni}). \tag{1.9}$$

You should convince yourself that (1) this is an antisymmetric $N \times N$ matrix and (2) any Hermitian antisymmetric $N \times N$ matrix can be written as a real linear combination of the $L_{(mn)}$'s. How many generators are there? There are $N(N-1)/2$ (you should also convince yourself of this).

The SO(N) Lie algebra is then given by computing the commutators of these matrices, which are

$$\left[L_{(mn)}, L_{(pq)}\right] = i\big(\delta_{mp}L_{(nq)} + \delta_{nq}L_{(mp)} - \delta_{np}L_{(mq)} - \delta_{mq}L_{(np)}\big). \tag{1.10}$$

1.2 U(N) and SU(N)

Another important Lie group is U(N), the group of $N \times N$ unitary matrices. It also has the important subgroup SU(N), consisting of those unitary matrices with determinant one.

Let's start with U(N), which is *defined* precisely as the type of faithful representation we need to get started. We already saw that if the infinitesimal transformation $U = 1 + i\alpha_a X_a$ is unitary, then X_a is Hermitian, and there are no further restrictions. So the generators of U(N) are simply Hermitian matrices X_a. Let's look at some special cases first.

Take $N = 1$, then any Hermitian matrix is just a real number, and we can take $X_1 = 1$. This is kind of boring in the same way the Lie algebra for SO(2) is boring. In fact, SO(2) and U(1) are isomorphic groups (convince yourself of this), so their Lie algebras also have to be the same.

$N = 2$ is more interesting, then there are four generators which we label as $X_0, \ldots, X_3$, and which are

$$X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.11}$$

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma^x \tag{1.12}$$

$$X_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{1}{2}\sigma^y \tag{1.13}$$

$$X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma^z, \tag{1.14}$$

where we have introduced the 2×2 Pauli matrices σ^x, σ^y and σ^z, and the factors of $1/2$ are purely conventional (but convenient). How do I know we have enough generators? Any 2×2 Hermitian matrix can be written as a linear combination of these matrices (convince yourself of this statement). We won't prove it, but a general U(2) matrix can be obtained by exponentiating these generators, $U = \exp(i\alpha_a X_a)$.

We can now specify the Lie algebra of SU(2) by the commutators

$$[X_0, X_a] = 0 \tag{1.15}$$

$$[X_i, X_j] = i\epsilon_{ijk} X_k, \tag{1.16}$$

where $a = 0, \ldots, 3$ and the indices $i, j, k = 1, \ldots, 3$. The first equation says X_0 commutes with everything else ... no surprise since it's the identity matrix. The second equation is the same as the Lie algebra for SO(3) – more on that later.

These commutation relations tell us that we can get a new Lie algebra by dropping X_0 entirely and just focusing on the three generators X_1, X_2 and X_3. Notice that these three generators are *traceless* Hermitian matrices. What Lie group do we get if we exponentiate *only* these generators, that is if we consider those unitary matrices with $U = \exp(i\alpha_i X_i)$? We get the group SU(2). Remembering that SU(2) is the group of unitary matrices with unit determinant, this follows from the same determinant identity Eq. (1.3), and the fact that the X_i are traceless.

More generally, U(N) has N^2 generators. Why N^2? This is because $N \times N$ Hermitian matrices are specified by N^2 real parameters – there are N real numbers giving the diagonal matrix elements, and there are $(N^2 - N)/2$ independent complex numbers above the diagonal, or equivalently $N^2 - N$ real numbers. (The matrix elements below the diagonal are given in terms of those above the diagonal by the requirement that the matrix is Hermitian.) So we have a total of N^2 real numbers.

As before, we can take one of the generators to be the identity matrix, and we can take the other $N^2 - 1$ to be some basis for traceless, Hermitian matrices. These $N^2 - 1$ matrices are the generators for SU(N). The difference between the U(N) and SU(N) Lie algebras is relatively trivial – at least, it's clear that if we can understand the SU(N) Lie algebra then we've understood the U(N) Lie algebra, too. This is why people generally focus on the SU(N) Lie algebra. Unlike for SO(N), we won't try to write down an explicit set of generators for SU(N), at least not yet.

2 Representations of a Lie algebra

Remember that a Lie algebra is the commutator algebra

$$[X_a, X_b] = i f_{abc} X_c. \tag{2.1}$$

While we introduced Lie algebras as commutator algebras of matrices, we can also think of the X_a's as some abstract objects satisfying the commutator algebra above. If one is really going to do that (we are not), then we have to more carefully define what we're talking about, and specify what properties these abstract objects have. Nonetheless it's helpful to be aware of the existence of such a more abstract point of view to have a sense of what we mean by representations of Lie algebras.

We know what it means to have a representation of a group. What does it mean to have a representation of a Lie algebra? A representation of a Lie algebra is a set of Hermitian matrices – we could call them M_a – with one matrix for each generator X_a, and satisfying the same commutator algebra, *i.e.*

$$[M_a, M_b] = i f_{abc} M_c. \tag{2.2}$$

Here we are not thinking abstractly – the M_a's are a bunch of matrices and we're looking at their commutators. Even if we obtained the X_a's as matrices, the M_a's could be different. For example, the M_a's can be matrices of a different size than the X_a's. If the matrices M_a are $d \times d$ matrices, we say the representation has dimension d.

Just as for representations of groups, we can talk about reducibility / irreducibility of representations of Lie algebras. A representation is reducible if it has an invariant subspace, which is just a subspace of column vectors that goes into itself upon multiplying by any of the matrices M_a. Irreducible reps (irreps) are those that are not reducible. Two representations M_a and M'_a are equivalent if there is a unitary matrix U such that $M'_a = U^\dagger M_a U$. (Since we take the M_a to be Hermitian, we don't want to consider a general similarity transformation.) A representation is completely reducible if one can go to a basis where the M_a's are block diagonal (every matrix M_a has to have the same block structure), with M_a matrices for irreps appearing in the blocks.

3 SU(2) vs. SO(3)

3.1 Double cover map

We've now seen that SU(2) and SO(3) have the same Lie algebra. But they are definitely *not* the same group. A quick way to see that is to observe that SU(2) has two elements that commute with all elements in the group, the 2×2 identity matrix I, and also $-I$, *i.e.* minus the identity matrix. By contrast, the only element of SO(3) that commutes with everything else is the identity matrix.

In fact there is a map from SU(2) to SO(3), which is said to give a "double cover" of SO(3). All this means is that there are exactly two elements of SU(2) that map to each element of SO(3). To get an idea of how this map works, first recall that any SO(3) matrix can be written

$$R = \exp(i\alpha_a L_a) = \exp(i\alpha\,\hat{\alpha}\cdot\vec{L}) \equiv R(\alpha,\hat{\alpha}), \tag{3.1}$$

where I arranged the three generators into a vector $\vec{L} = (L_1, L_2, L_3)$, and I am also thinking of α_a as a vector $\vec{\alpha} = \alpha\hat{\alpha}$, where $\alpha = |\vec{\alpha}|$ and $\hat{\alpha}$ is a unit vector. The interpretation here is that R is a rotation by the angle α about the axis given by $\hat{\alpha}$. Note that rotation by 2π is the same as no rotation at all; that is $R(2\pi,\hat{\alpha}) = I$.

Suggestively, we can also write a general SU(2) matrix as

$$U = \exp\left(\frac{i\alpha}{2}\hat{\alpha}\cdot\vec{\sigma}\right) \equiv U(\alpha,\hat{\alpha}). \tag{3.2}$$

But now, $U(2\pi,\hat{\alpha}) = -I$. In fact, more generally, $U(\alpha+2\pi,\hat{\alpha}) = -U(\alpha,\hat{\alpha})$. This suggests that we have a two-to-one map

$$U(\alpha,\hat{\alpha}) \mapsto R(\alpha,\hat{\alpha}), \tag{3.3}$$

where clearly $U(\alpha,\hat{\alpha})$ and $U(\alpha+2\pi,\hat{\alpha})$ both map to the same SO(3) rotation matrix.

There is indeed such a map, and let's describe how to obtain it, then give an explicit formula. First of all I claim that, given any $U \in \mathrm{SU}(2)$, the following equation holds:

$$U\sigma^i U^\dagger = R_{ji}(U)\sigma^j. \tag{3.4}$$

Here $i, j = x, y, z$ and there is an implied sum on the right hand side (as usual). (Note there is a different placement of the indices from how I originally defined this in lecture – the reason I changed conventions is to make things work out nicely later on.) This equation is saying that $U\sigma^i U^\dagger$ is a linear combination of Pauli matrices, with coefficients $R_{ij}(U)$. (It's clear that these coefficients are a function of U, because U is the only information we specify on the left-hand side.) How to see that this is true. First, it is obvious that $U\sigma^i U^\dagger$ is Hermitian (just take the Hermitian conjugate). Second,

$$\mathrm{tr}(U\sigma^i U^\dagger) = \mathrm{tr}(\sigma^i U^\dagger U) = \mathrm{tr}(\sigma^i) = 0, \tag{3.5}$$

so $U\sigma^i U^\dagger$ is both traceless and Hermitian. But any traceless, Hermitian 2×2 matrix can be written as a linear combination of the Pauli matrices with *real* coefficients. Therefore Eq. (3.4) is true, and $R_{ji}(U) \in \mathbb{R}$.

So far, from U we have obtained a set of 9 real numbers, $R_{ji}(U)$, which we can assemble into a 3×3 matrix $R(U)$. We still need to show that $R(U)$ is a rotation matrix, *i.e.* that it is an element of SO(3). To do this we will use the Pauli matrix identity

$$\sigma^i\sigma^j = \delta_{ij} I_{2\times 2} + i\epsilon_{ijk}\sigma^k, \tag{3.6}$$

where $I_{2\times 2}$ is the 2×2 identity matrix. Then we will find two different expressions for $U\sigma^i\sigma^j U^\dagger$ and compare them. First,

$$U\sigma^i\sigma^j U^\dagger = U[\delta_{ij} I_{2\times 2} + i\epsilon_{ijk}\sigma^k]U^\dagger = \delta_{ij} I_{2\times 2} + i\epsilon_{ijk} U\sigma^k U^\dagger. \tag{3.7}$$

The second term is a linear combination of Pauli matrices, but we won't need to simplify it further. Alternatively we have

$$\begin{aligned} U\sigma^i\sigma^j U^\dagger &= (U\sigma^i U^\dagger)(U\sigma^j U^\dagger) = R_{ki}(U)R_{\ell j}(U)\sigma^k\sigma_\ell = R_{ki}(U)R_{\ell j}(U)[\delta_{k\ell}I_{2\times 2} + i\epsilon_{k\ell m}\sigma^m] \\ &= R_{ki}(U)R_{kj}(U)I_{2\times 2} + \cdots = [R(U)^T R(U)]_{ij} I_{2\times 2} + \cdots, \end{aligned} \tag{3.8}$$

where the $\cdots$ is a linear combination of Pauli matrices that we don't need to examine. Setting these two results equal, we have $[R(U)^T R(U)]_{ij} = \delta_{ij}$, or equivalently $R(U)^T R(U) = I_{3\times 3}$, so $R(U)$ is an orthogonal matrix, i.e. $R(U) \in \mathrm{O}(3)$.

We still need to show $R(U) \in \mathrm{SO}(3)$; at this point all we need to show is that $\det R(U) = 1$. Rather than showing this directly, we can simply observe that since $U = \exp(i\vec{\alpha}\cdot\vec{\sigma}/2)$, we can continuously deform U to I simply by making $\vec{\alpha}$ smaller and smaller. Since R is a continuous function of U, and the determinant is a continuous function, $\det R(U)$ cannot jump discontinuously as we change U in this way. Since the only two possible values of $\det R(U)$ for an O(3) matrix are ± 1, and since $\det I = 1$, we must have $\det R(U) = 1$.

Now in principle we are done – we've shown that Eq. (3.4), given $U \in SU(2)$, produces for us $R(U) \in SO(3)$. It's clear that $R(U) = R(-U)$, because changing $U \to -U$ does not affect the right-hand side of Eq. (3.4). It's nice to go a bit further and obtain an explicit formula for $R(U)$ in terms of U. We can do this by multiplying both sides of Eq. (3.4) by a Pauli matrix $\sigma^k/2$, and taking the trace. We have

$$\frac{1}{2}\,\mathrm{tr}\left[\sigma^k U\sigma^i U^\dagger\right] = \frac{1}{2}R_{ji}(U)\,\mathrm{tr}\left[\sigma^k\sigma^j\right] = R_{ji}(U)\delta_{kj} = R_{ki}(U), \tag{3.9}$$

where we used the fact that $\mathrm{tr}[\sigma^i\sigma^j] = 2\delta_{ij}$. Rewriting this a bit more cleanly gives us the formula we want:

$$R_{ij}(U) = \frac{1}{2}\,\mathrm{tr}\left[\sigma^i U\sigma^j U^\dagger\right]. \tag{3.10}$$

3.2 Representations and projective representations

Now let's discuss a different aspect of the relationship between SU(2) and SO(3). Because the Lie algebras are the same, it is obvious that both the matrices L_a, and the matrices $\sigma^i/2$, give representations of both Lie algebras. But what about representations of the two groups? Obviously we have (1) a two-dimensional rep of SU(2) with matrices $\exp(i\vec{\alpha}\cdot\vec{\sigma}/2)$, and (2) a three-dimensional rep of SO(3) with matrices $\exp(i\alpha_a L_a)$.

First, can we think about (2) as a rep of SU(2)? The answer is yes, and this is just an application of the double-cover map. The group elements of SU(2) are 2×2 unitary matrices U, and the double cover map gives us $R(U) \in SO(3)$. To show this is a representation, we need to show $R(U_1)R(U_2) = R(U_1U_2)$. We can show this by evaluating $U_1U_2\sigma^i U_2^\dagger U_1^\dagger$ in two different ways. First we have

$$U_1U_2\sigma^i U_2^\dagger U_1^\dagger = (U_1U_2)\sigma^i(U_1U_2)^\dagger = R_{ji}(U_1U_2)\sigma^j. \tag{3.11}$$

Or instead we can write

$$U_1 U_2 \sigma^i U_2^\dagger U_1^\dagger = U_1 (U_2 \sigma^i U_2^\dagger) U_1^\dagger = U_1 [R_{ji}(U_2)\sigma^j] U_1^\dagger$$
$$= R_{ji}(U_2) R_{kj}(U_1) \sigma^k = [R(U_1)R(U_2)]_{ki}\sigma^k. \quad (3.12)$$

Setting these equal gives the desired result.

But can we think of (1) as a rep of SO(3)? The answer is "not quite," but we can if we generalize a bit what we mean by a representation. We need a function that, given $R \in SO(3)$, outputs $U(R) \in \mathrm{SU}(2)$. The problem is that there are two natural choices for each R; we can take either $U \in \mathrm{SU}(2)$ that maps into R under the double cover map. That is, if we choose $U(R)$, then $-U(R)$ is an equally good choice. But ok, let's make some choice arbitrarily.

The next thing to do is to understand what happens when we multiply $U(R_1)$ and $U(R_2)$. It turns out that

$$U(R_1)U(R_2) = \omega(R_1, R_2) U(R_1 R_2), \quad (3.13)$$

where $\omega(R_1, R_2) = \pm 1$. For example, take $R_1 = R_2 = R(\pi, \hat{z})$, a rotation by π about the z-axis. Then $R_1 R_2 = R_1^2 = I$. Taking $U(R_1) = U(R_2) = \exp(i\pi\sigma^z/2)$, we have

$$U(R_1)U(R_2) = \exp(i\pi\sigma^z) = -I = (-1)U(R_1 R_2). \quad (3.14)$$

Note that we still get the -1 even if we made the other choice for $U(R_1)$.

Therefore what we have is not really a representation. But we can say the $U(R)$ matrices form a representation "up to a phase factor" $\omega(R_1, R_2)$ – that is, multiplying the $U(R)$'s is almost compatible with the multiplication table of the group. There are "mistakes," but the mistakes only result in being off by a phase factor. So we have a kind of generalized representation, which is referred to as a *projective representation*. We can thus say that we have a two-dimensional projective representation of SO(3), given by the matrices $U(R)$. In quantum mechanics, projective representations show up all the time and are quite reasonable, because the overall phase of a quantum state is not observable anyway. Of course the most familiar projective representation in physics is the same one we've been discussing, which is often referred to as the spin-1/2 representation of SO(3).

4 Irreducible representations of the SO(3) / SU(2) Lie algebra

In quantum mechanics courses, you learn about the irreps of the SO(3) / SU(2) Lie algebra, under the heading of theory of angular momentum. Therefore I don't want to dwell on how to find these irreps and know that you've found them all; instead I just want to list them and describe some of their properties. You can find more details in Zee IV.2 or in your favorite quantum mechanics book.

I'll refer to the Lie algebra in question as the SO(3) Lie algebra, and denote the generators by J_1, J_2, J_3, which obey the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k. \quad (4.1)$$

The irreps are labeled by a single parameter j, which takes values $j = 0, 1/2, 1, 3/2, \dots$. Let's pick a value of j and describe the corresponding irrep. There are orthonormal basis states $\{|j,m\rangle\}$, where $m = -j, -j+1, \dots, +j$. There are clearly $2j+1$ such basis states, and the dimension of the representation is $2j+1$. This representation is sometimes called the "spin-j" representation, especially when its physical origin is from electron spin, but sometimes even when the states originate from some other degrees of freedom (*e.g.* orbital angular momentum). We have to say how the operators J_i act on the basis states. We have

$$J_3|j,m\rangle = m|j,m\rangle; \tag{4.2}$$

that is, the basis states are eigenstates of the J_3 generator. It's easiest to write down the action of J_1 and J_2 by defining the raising and lowering operators $J_\pm = J_1 \pm iJ_2$, and these act on the basis states by

$$J_+|j,m\rangle = \sqrt{(j+1+m)(j-m)}|j,m+1\rangle, \tag{4.3}$$

and

$$J_-|j,m\rangle = \sqrt{(j+1-m)(j+m)}|j,m-1\rangle. \tag{4.4}$$

From these equations it is straightforward to show that

$$\vec{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \tag{4.5}$$

where $\vec{J}^2 = J_1^2 + J_2^2 + J_3^2$. It follows (or can be shown directly) that $[\vec{J}^2, J_i] = 0$. In the context of Lie algebras, this operator, whose eigenvalue tells us which irrep we are looking at, is referred to as a Casimir invariant for the SO(3) Lie algebra. Other Lie algebras also have Casimir invariants that commute with the generators and whose eigenvalue only depends on which irrep one is looking at. Some Lie algebras have multiple Casimirs, but SO(3) only has one, and in the SO(3) case, the Casimir eigenvalue uniquely tells us which irrep we are looking at.

5 Adjoint representation

Every Lie algebra has a representation called the adjoint representation (and it turns out to be an irrep at least for most of the Lie algebras we are considering). The way we construct the adjoint representation is analogous to the way we got the double cover map from SU(2) to SO(3). In fact our construction will give us the adjoint representation of the Lie group, and then we will get the adjoint rep of the Lie algebra by taking infinitesimal transformations.

Consider a Lie algebra with N matrix generators X_a and commutation relations $[X_a, X_b] = if_{abc}X_c$. Let $U = \exp(i\alpha_b X_b)$, then I claim that

$$UX_aU^\dagger = R_{ba}(U)X_b, \tag{5.1}$$

where $R(U)$ is a real $N \times N$ matrix, and moreover $R(U_1)R(U_2) = R(U_1U_2)$, *i.e.* $R(U)$ gives an N-dimensional representation of the Lie group.

To see this claim is true, let's consider a matrix-valued function

$$f(\lambda) = e^{\lambda A} B e^{-\lambda A}, \tag{5.2}$$

where A and B are square matrices and λ is a complex number. We can understand the Taylor series for $f(\lambda)$ about $\lambda = 0$ by computing derivatives. First we have $f(0) = B$. Next,

$$\frac{df}{d\lambda} = Af(\lambda) - f(\lambda)A = [A, f(\lambda)]. \tag{5.3}$$

Therefore the nth derivative is an n-fold nested commutator, for instance the second derivative is

$$\frac{d^2 f}{d\lambda^2} = [A, [A, f(\lambda)]]. \tag{5.4}$$

We thus have the Taylor series

$$f(\lambda) = B + \lambda[A, B] + \frac{\lambda^2}{2}[A, [A, B]] + \frac{\lambda^3}{3!}[A, [A, [A, B]]] + \cdots \tag{5.5}$$

We can then apply this result to Eq. (5.1) with $B = X_a$ and $A = i\alpha_b X_b$, to see that $UX_aU^\dagger$ is also a series of nested commutators. The first two terms are

$$UX_aU^\dagger = X_a + i[\alpha_b X_b, X_a] + \cdots = X_a - \alpha_b f_{bac} X_c + \cdots, \tag{5.6}$$

and we see that we have a real linear combination of the generators. (It is not hard to convince yourself this conclusion holds up to arbitrarily high order.) So we've shown that $R(U)$ is a real $N \times N$ matrix.

Next, showing $R(U_1)R(U_2) = R(U_1U_2)$ proceeds by exactly the same argument as when we showed this for the double cover map. We won't repeat it again because it's really exactly the same.

Now that we have a real N-dimensional rep of the Lie group, let's consider infinitesimal transformations and get the corresponding N-dimensional rep of the Lie algebra. We let $U = I + i\alpha_b X_b$, and then $R_{ab}(U) = \delta_{ab} + i\alpha_c (T^c)_{ab}$, where the matrices T^c are the generators of the adjoint rep that we'd like to find. Plugging this into Eq. (5.1) we get

$$(I + i\alpha_b X_b)X_a(I - i\alpha_c X_c) = X_a + i\alpha_c (T^c)_{ba} X_b \tag{5.7}$$

Expanding the left-hand side to first order in α we have

$$(I + i\alpha_b X_b)X_a(I - i\alpha_c X_c) = X_a + i[\alpha_b X_b, X_a] = X_a - \alpha_b f_{bac} X_c = X_a - \alpha_c f_{cab} X_b \tag{5.8}$$

Matching these two results gives

$$(T^c)_{ba} = if_{cab} = -if_{acb}. \tag{5.9}$$

So we've found a representation of the Lie algebra whose matrices are given by the structure constants. It's actually not obvious from what we've done that these matrices are Hermitian (and the corresponding representation of the Lie group is unitary). The right statement is that it's possible to choose a basis for the generators X_a that makes the T^c matrices Hermitian, which also implies that in this cases $f_{abc} = -f_{acb}$. Since the structure constants are already antisymmetric in the first two indices, making them anti symmetric in the second two indices makes them completely antisymmetric! Often people choose the structure constants to be completely antisymmetric.

6 SO(N) tensors

In terms of applications, so far we've mainly thought about representations as coming from the action of symmetry on the Hilbert space of a quantum system. For instance, irreps arise as multiplets of degenerate energy eigenstates. But this is not the only way that representations show up in quantum systems – observables can also form representations of a symmetry group. Understanding how observables transform under symmetry is important *e.g.* for selection rules which we will come to later. At the same time, thinking along these lines we will find a large family of irreps of SO(N).

A familiar example is the position operator x_i ($i = 1, 2, 3$) of a quantum particle in three dimensions. Of course, SO(3) rotation is a symmetry of many familiar systems in single-particle quantum mechanics, *e.g.* the hydrogen atom. Under a rotation $R \in \mathrm{SO}(3)$, the position operator changes by

$$x_i \to R_{ij} x_j. \tag{6.1}$$

We can express this situation by saying "x_i transforms as a representation of SO(3)." To be a bit pedantic, we can think of x_i as a three-component column vector and write the transformation Eq. (6.1) as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{6.2}$$

My point in writing things this way is just to make it clear that we are thinking of x_i as a column vector in a three-dimensional vector space, and R is a 3×3 matrix acting on these column vectors

We can also express this situation by saying that x_i is a SO(3) tensor. In general, a SO(N) tensor is just a representation where the "column vectors" are objects carrying some number of indices, and each index gets rotated independently of the others using a matrix $R \in \mathrm{SO}(N)$. If that statement does not make sense yet, that's fine, it's easier to understand what we mean by a tensor by looking at some examples.

A particularly simple SO(N) tensor (or, we can also say SO(N) tensor rep) is the *vector* v_i ($i = 1, \ldots, N$). For example, v_i could be the position operator of a particle moving in N-dimensional space. Under a SO(N) rotation $R \in \mathrm{SO}(N)$, v_i transforms by $v_i \to R_{ij} v_j$. Clearly we can think of v_i as an N-component column vector, and v_i transforms as an N-dimensional representation rep of SO(N). Of course this is just the familiar rep of SO(N) in terms of $N \times N$ SO(N) matrices. We often refer to this rep as "the vector rep of SO(N)."

The vector rep is irreducible for $N > 2$. For $N = 3$, note that the vector rep is the same as the $j = 1$ rep, and also the same as the adjoint rep. Remember that the dimension of the adjoint rep is always the same as the number of generators, which is $N(N-1)/2$ for SO(N), and this happens to equal 3 when we put $N = 3$. But for $N > 3$, the dimension of the adjoint rep is larger than N, and the adjoint rep is not the same as the vector rep.

Now let's move on to something a little more interesting. We consider a 2 index tensor T_{ij}, where $i, j = 1, \ldots, N$. There are N^2 different choices for the pair of indices i and j, and this tensor has N^2 independent components. That is, thinking of each T_{ij} as a real number, we have a set of N^2 independent real numbers. Under $R \in \mathrm{SO}(N)$, this tensor transforms by

$$T_{ij} \to T'_{ij} = R_{ik} R_{j\ell} T_{k\ell}. \tag{6.3}$$

This is what we mean when we say "each index gets rotated independently of the others." Note that without Eq. (6.3), it would not be correct to say T_{ij} is a tensor. When we say something is a tensor, we do not only mean that it some object with a bunch of indices. We also imply something about how those indices transform under some symmetry group.

A quick point about terminology: we say T_{ij} is a rank-2 tensor, because it has two indices. The vector v_i is a rank-1 tensor. The rank of a $\mathrm{SO}(N)$ tensor is just the number of its indices. By the way, there is a rank-0 tensor, the scalar representation. This tensor has a single component s, which does not change under a rotation, $s \to s$. The scalar rep is the same as the one-dimensional trivial rep.

It was pretty obvious that the vector gives us a representation of $\mathrm{SO}(N)$, thinking of v_i as a column vector. But looking at Eq. (6.3) this may be less obvious. In fact, T_{ij} transforms as a N^2-dimensional rep of $\mathrm{SO}(N)$. We can rewrite Eq. (6.3) by thinking about T_{ij} as forming a big column vector with N^2 elements, and it gets multiplied by a big $N^2 \times N^2$ matrix whose entries are products of matrix elements of R. That is,

$$\begin{pmatrix} T'_{11} \\ T'_{12} \\ T'_{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} R_{11}R_{11} & R_{11}R_{12} & \cdots \\ R_{11}R_{21} & R_{11}R_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \\ \vdots \end{pmatrix}. \tag{6.4}$$

This makes it clear that we can think of T_{ij} as giving us a N^2-dimensional rep of $\mathrm{SO}(N)$.

Actually, if we want to be careful we should check that the matrices of this N^2-dimensional rep satisfy the usual rule $D(g_1)D(g_2) = D(g_1 g_2)$. To do that, consider two rotations $R^1, R^2 \in \mathrm{SO}(N)$. We consider two ways of transforming T_{ij}: (1) We first transform by R^2, and then by R^1. (2) We transform all at once using the product $R^1 R^2 \in \mathrm{SO}(N)$. If these two ways of transforming T_{ij} agree, then we in fact have a representation. Let's do (1). We have

$$T_{ij} \to T'_{ij} = R^2_{ik} R^2_{j\ell} T_{k\ell}, \tag{6.5}$$

followed by

$$T'_{ij} \to T''_{ij} = R^1_{ik} R^1_{j\ell} T'_{k\ell} = R^1_{ik} R^2_{km} R^1_{j\ell} R^2_{\ell n} T_{mn} = (R^1 R^2)_{im} (R^1 R^2)_{jn} T_{mn}. \tag{6.6}$$

This is the same as what we get if we do (2).

The next question to ask is whether the rank-2 tensor T_{ij} is irreducible. It turns out that it is reducible. We can see this by writing T_{ij} as a sum of symmetric and antisymmetric tensors. We define

$$T^S_{ij} = \frac{1}{2}\big(T_{ij} + T_{ji}\big), \tag{6.7}$$

which is clearly a symmetric rank-2 tensor, *i.e.* $T^s_{ij} = T^s_{ji}$. We also define

$$T^A_{ij} = \frac{1}{2}\big(T_{ij} - T_{ji}\big), \tag{6.8}$$

which satisfies $T^A_{ij} = -T^A_{ji}$; that is, this is an antisymmetric tensor. We have

$$T_{ij} = T^S_{ij} + T^A_{ij}. \tag{6.9}$$

We need to check that T^S_{ij} and T^A_{ij} actually transform like tensors under SO(N). Let's look at T^S_{ij}. We have

$$\begin{aligned} T^S_{ij} &= \frac{1}{2}\big(T_{ij} + T_{ji}\big) && (6.10)\\ &\to \frac{1}{2}\big(R_{ik}R_{j\ell}T_{k\ell} + R_{j\ell}R_{ik}T_{\ell k}\big) && (6.11)\\ &= R_{ik}R_{j\ell}T^S_{k\ell}, && (6.12)\end{aligned}$$

so indeed T^S_{ij} is an SO(N) tensor. (Note that I chose the names for the dummy indices judiciously on the second line – if I didn't make a nice choice in advance, I could have always just renamed dummy indices as needed.)

It is easy to check that the same result holds for the antisymmetric tensor, that is

$$T^A_{ij} \to R_{ik}R_{j\ell}T^A_{k\ell}. \tag{6.13}$$

What we've seen so far is that the symmetric tensor goes into itself under SO(N) transformations, and similarly for the antisymmetric tensor. This implies that the tensor T_{ij} is a reducible rep – we have reduced it into its symmetric and antisymmetric parts. Thinking of T_{ij} as a N^2-component column vector again, writing T_{ij} in terms of symmetric and antisymmetric parts gives a change of basis, where now in our column vector we first list all the independent components of T^S_{ij} in the column vector, then we list all the independent components of T^A_{ij}:

$$\begin{pmatrix} T^S_{11} \\ T^S_{12} \\ \vdots \\ \hline T^A_{12} \\ T^A_{13} \\ \vdots \end{pmatrix} \tag{6.14}$$

Because the symmetric and antisymmetric parts of T_{ij} don't "mix" under SO(N) transformations, our discussion above implies

$$\begin{pmatrix} T^S_{11} \\ T^S_{12} \\ \vdots \\ \hline T^A_{12} \\ T^A_{13} \\ \vdots \end{pmatrix} \to \left(\begin{array}{c|c} \mathcal{R}^S & 0 \\ \hline 0 & \mathcal{R}^A \end{array}\right) \begin{pmatrix} T^S_{11} \\ T^S_{12} \\ \vdots \\ \hline T^A_{12} \\ T^A_{13} \\ \vdots \end{pmatrix}, \tag{6.15}$$

where $\mathcal{R}^S$ and $\mathcal{R}^A$ are square matrices encoding the SO(N) transformations of the symmetric and antisymmetric parts, respectively. Writing things out this way makes it clear that we have reduced the T_{ij} tensor rep.

6.1 Rank-2 antisymmetric tensor

It turns out that the antisymmetric tensor T^A_{ij} is an irrep – we won't try to prove this. In fact, this is the same as the adjoint rep. Remember that for SO(N), the adjoint rep is given by

$$X_a \to R X_a R^T, \tag{6.16}$$

where the X_a are the imaginary antisymmetric matrices generating SO(N), which should be thought of as basis vectors for the adjoint rep. A general *real* antisymmetric matrix T^A can be written as a linear combination

$$T^A = i \sum_a t_a X_a, \tag{6.17}$$

where t_a are real parameters. Therefore we can think of T^A as transforming in the adjoint rep by

$$T^A \to i \sum_a t_a R X_a R^T = R T^A R^T. \tag{6.18}$$

But this is exactly how the tensor T^A_{ij} transforms if we view it as a $N \times N$ matrix; we have

$$T^A_{ij} \to R_{ik} R_{j\ell} T^A_{k\ell} = R_{ik} T^A_{k\ell} R_{j\ell} = R_{ik} T^A_{k\ell} R^T_{\ell j} = (R T^A R^T)_{ij}. \tag{6.19}$$

For $N = 3$ something special happens with T^A_{ij}. We can contract the antisymmetric tensor with the epsilon tensor to get $\epsilon_{ijk} T^A_{jk}$. It looks like – and it can be checked – this transforms like the vector rep. So for $N = 3$ the rank-2 antisymmetric tensor is the same as the vector (and it's also the same as the adjoint rep). Another way to understand this is to make the familiar observation that if $\vec{v}$ and $\vec{w}$ are two SO(3) vectors, then the cross product $\vec{v} \times \vec{w}$, which is an antisymmetric combination of the two vectors, is another SO(3) vector. In equations,

$$(\vec{v} \times \vec{w})_i = \epsilon_{ijk} v_j w_k = \frac{1}{2} \epsilon_{ijk} (v_j w_k - v_k w_j). \tag{6.20}$$

6.2 Rank-2 symmetric tensor

It turns out that the rank-2 symmetric tensor T^S_{ij} is reducible. To see this, we observe that we can make a scalar rep out of T^S_{ij}, by contracting it with δ_{ij}. That is we have

$$\delta_{ij} T^S_{ij} \to \delta_{ij} R_{ik} R_{j\ell} T^S_{k\ell} = (R^T R)_{k\ell} T^S_{k\ell} = \delta_{ij} T^S_{ij}. \tag{6.21}$$

Contracting δ_{ij} with T^S_{ij} is called "taking the trace" of T^S_{ij}, and of course this is the same as taking the trace if we view T^S_{ij} as a matrix. This terminology is used a bit more generally with tensors, though, as we'll see below.

To completely reduce T^S_{ij}, we need to write it as a linear combination of a scalar part, and a traceless part. The traceless part is

$$\tilde{T}^S_{ij} = T^S_{ij} - \frac{1}{N}\delta_{ij}\delta_{k\ell}T^S_{k\ell}, \tag{6.22}$$

and it can be checked that $\delta_{ij}\tilde{T}^S_{ij} = 0$. It turns out that this is an irreducible tensor. We can write T^S_{ij} as the linear combination

$$T^S_{ij} = \tilde{T}^S_{ij} + \frac{N-1}{N}\delta_{ij}\delta_{k\ell}T^S_{k\ell}, \tag{6.23}$$

where the second term transforms like a scalar.

To summarize, now we have completely reduced the rank-2 tensor T_{ij}, which decomposes into three irreducible tensors: the rank-2 antisymmetric tensor, the rank-2 traceless symmetric tensor, and a scalar.

For $N = 3$, it turns out that $\tilde{T}^S_{ij}$ is the same as the $j = 2$ irrep. One way to guess this is true is to count the number of independent components (see below), and observe there are five components, which is the same as $2j + 1 = 2 \cdot 2 + 1 = 5$. (To prove this, we could think in terms of angular momentum addition, and observe that we get a spin-2 upon taking a symmetric combination of two spin-1's. But we aren't going to discuss this for lack of time.)

6.3 Higher-rank tensors

We can keep going with similar ideas to construct higher-rank irreducible tensors. For instance a rank-3 completely antisymmetric tensor T^A_{ijk} is irreducible. "Completely antisymmetric" means that we get a minus sign upon exchanging any pair of indices, *e.g.*

$$T^A_{ijk} = -T^A_{jik} = -T^A_{ikj} = -T^A_{kji}. \tag{6.24}$$

Similar to the rank-2 case, a rank-3 completely symmetric tensor T^S_{ijk} is not irreducible, because we need to "subtract out the trace." First of all, for T^S_{ijk} to be completely symmetric means

$$T^S_{ijk} = T^S_{jik} = T^S_{ikj} = T^S_{kji}, \tag{6.25}$$

from which it follows that we can permute the indices however we want. But the trace $\delta_{ij}T^S_{ijk}$ transforms like the vector rep, and we'd like to subtract this out to get a rank-3 traceless symmetric tensor $\tilde{T}^S_{ijk}$ satisfying $\delta_{ij}\tilde{T}^S_{ijk} = 0$. (Notice that while we took the trace on the first two indices, this doesn't matter – because T^S_{ijk} is symmetric, we will get the same tensor no matter which two indices we trace over.) The irreducible tensor we want is given by the following expression:

$$\tilde{T}^S_{ijk} = T^S_{ijk} - \frac{1}{N+2}\big(\delta_{ij}\delta_{\ell m}T^S_{\ell mk} + \delta_{ik}\delta_{\ell m}T^S_{\ell mj} + \delta_{jk}\delta_{\ell m}T^S_{\ell mi}\big). \tag{6.26}$$

Something different from the rank-2 case is what happens if we try to reduce a rank-3 tensor T_{ijk} with no symmetry properties what soever, here we have

$$T_{ijk} = T^S_{ijk} + T^A_{ijk} + \cdots, \tag{6.27}$$

where the "$\cdots$" is non-zero, and includes tensors with definite patterns of symmetry of the indices which are neither completely symmetric nor completely antisymmetric. Indeed, to find all the irreducible tensors at rank-3 and above we would need to consider such tensors of more complicated symmetry, but we won't do that in this course.

6.4 Counting independent components

Now we've that we've discussed a number of tensor reps of SO(N), including a number of irreducible tensors, you might wonder what are the dimensions of all these reps. Put another way, how many independent components do all these tensors have? Finding the dimensions of irreps (or, equivalently, counting the number of independent components of an irreducible tensor) is important in physics applications. Here is one reason this is important: in particle physics, when one has a Lie group symmetry G, irreps of G can correspond to multiplets of particles all with the same mass (or, if the symmetry is approximate, with approximately the same mass). Group theory has been used to explain such multiplets of particles. At the same time, experimentally finding and then understanding such multiplets in terms of symmetry has shed light on symmetries (and approximate symmetries) in particle physics, and this has helped to constrain the development of theories and lead to further predictions. The importance of this role of group theory in particle physics is even built into the common physicist's notation of referring to irreps by their dimensions.