# Born Approximation

Need to compute $f(\vec{k}', \vec{k})$

$\rightarrow$ Need $\langle \vec{k}' | V | \psi^+ \rangle = \langle \vec{k}' | T | \vec{k} \rangle$

$$T = V + V \frac{1}{E - H_0 + i\varepsilon} V + \ldots$$

gives $f = f^{(1)} + f^{(2)} + \ldots$

$$f^{(1)}(\vec{k}', \vec{k}) \equiv -\frac{m}{2\pi\hbar^2} \underbrace{\int d^3\vec{r}\,' \, e^{i(\vec{k} - \vec{k}') \cdot \vec{r}\,'} \, V(\vec{r}\,')}_{\text{just Fourier transform } \tilde{V}(\vec{q})\text{; } \vec{q} = \vec{k}' - \vec{k}.}$$

If $V$ spherically symmetric, $\tilde{V}(\vec{q}) = \tilde{V}(q)$; $q = |\vec{q}|$.

$$q = |\vec{k} - \vec{k}'| = \sqrt{2k^2 - 2kk\cos\theta}$$

$$= \sqrt{2}\, k \sqrt{1 - \cos\theta}$$

$$= \sqrt{2} \cdot \sqrt{2}\, k \sqrt{\frac{1 - \cos\theta}{2}} = 2k \sin\frac{\theta}{2}.$$

In this case,

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{m}{2\pi\hbar^2}\cdot 2\pi \int_0^\infty dr\, r^2\, V(r) \underbrace{\int_{-1}^{1} d(\cos\theta)\, e^{-iqr\cos\theta}}_{\substack{= \frac{-1}{iqr}\left[e^{-iqr} - e^{iqr}\right] \\ = \frac{2\sin qr}{qr}}}$$

$$= -\frac{2m}{\hbar^2 q}\int_0^\infty dr\, r\, V(r)\, \sin(qr).$$

~~Small~~

Example: Yukawa Potential $V(r) = \dfrac{V_0 e^{-\mu r}}{\mu r}$.

Note: If take $\frac{V_0}{\mu} \longrightarrow$ const ~~to const~~ and $\mu \to 0$, then this becomes the Coulomb potential.

$$f^{(1)}(\vec{k}', \vec{k}) = -\frac{2mV_0}{\mu\hbar^2 q}\int_0^\infty dr\, e^{-\mu r}\sin(qr).$$

To evaluate:

$$\int_0^\infty dr\, e^{-\mu r}\sin(qr) = \mathrm{Im}\left[\int_0^\infty e^{-\mu r} e^{iqr}\, dr\right].$$

$$= \mathrm{Im}\left[-\frac{1}{iq-\mu}\right] = \frac{q}{\mu^2+q^2}$$

$$\Rightarrow f^{(1)}(q) = -\left(\frac{2mV_0}{\mu\hbar^2}\right)\frac{1}{q^2+\mu^2}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{2mV_0}{\mu\hbar^2}\right)^2\frac{1}{(q^2+\mu^2)^2} \quad ; \quad q^2 = 4k^2\sin^2\frac{\theta}{2}$$

$$= 2k^2(1-\cos\theta).$$

$\cancel{q^2 = 4k^2\sin^2\frac{\theta}{2}}$

Take $\mu \to 0$ with $\frac{V_0}{\mu} \to e^2$, then

$$\left(\frac{d\sigma}{d\Omega}\right) \to \frac{(2m)^2 e^4}{\hbar^4} \frac{1}{16 k^4 \sin^4(\frac{\theta}{2})}.$$

$$= \frac{1}{16}\left(\frac{e^2}{E}\right)^2 \frac{1}{\sin^4(\frac{\theta}{2})}$$

Same as classical Rutherford scattering.

## Validity of Born Approximation

~~Scatt~~

Wavefunction: $\psi^+(\vec{r}) \approx \frac{1}{L^3}\left[ e^{i\vec{k}\cdot\vec{r}} + \frac{e^{ikr}}{r} f(q) \right]$.

$\Rightarrow$ Expect $\frac{f(q)}{a} \ll 1$ for some length $a$.

What's a? $a \sim$ range of potential (smallest length beyond which ~~scattering form~~ asymptotic form of $\psi^+$ is valid) really means $ka \ll 1$.

At <u>low energy</u>:

$$f(q) = -\frac{2m}{\hbar^2 q}\int_0^\infty dr\, r V(r) \sin(qr)$$

"typical value of $V(r)$"

$$\approx -\frac{2m}{\hbar^2}\int_0^\infty dr\, r^2 V(r) \approx -\frac{2m a^3 V_0}{\hbar^2}$$

$$\Rightarrow \frac{f(q)}{a} \ll 1 \Rightarrow \left|\frac{m a^2 V_0}{\hbar^2}\right| \ll 1$$

~~High energy~~

~~High energy: sin(qr) oscillates rapidly at large r~~

$$\cancel{\int_0^\infty dr\, r V(r) \sin(qr) \approx q \int_0^{1/q} dr\, r^2 V(r) \approx \frac{1}{q^2} V_0}$$