Physics 7240: Advanced Statistical Mechanics

Lecture 4: Field Theory Primer

Leo Radzihovsky

(Dated: October 11, 2017)

Abstract

In these lecture notes, starting with a review of ordinary Gaussian integrals, we will develop the *functional* integral calculus. We will then utilize it to go beyond Landau mean-field theory by including fluctuations over coarse-grained fields and thereby obtaining a complete statistical mechanics analysis. After warming up on non-interacting systems, described by harmonic Hamiltonians, we will treat interacting, nonlinear systems using perturbative expansion in nonlinearities. We will discover a breakdown of this perturbative expansion near critical points of continuous phase transitions, thereby finding regime of breakdown of Landau mean-field theory, i.e., the Ginzburg region. These developments will set stages for our later renormalization group and phase stability studies.

- Gaussian integrals
- Functional integrals
- Perturbative diagrammatic expansion
- Breakdown of perturbation theory and Ginzburg criterion

I. INTRODUCTION AND MOTIVATION

It is not too strong of an exaggeration to say that our theoretical capabilities are limited to a harmonic oscillator, $H = \frac{1}{2}p^2/m + \frac{1}{2}m\omega_0^2 x^2$, — and systems related to it by a clever transformation and/or perturbative expansion. With some ingenuity, this still covers a rich variety of systems, allowing impressive progress in physics. This is not surprising, as stable states of a system occur at minima of a Hamiltonian, around which, at low energies the Hamiltonian (as any function) is harmonic, with only small nonlinear corrections. It is thus imperative to have a thorough understanding of a harmonic oscillator.

In statistical mechanics, the main object is the partition function,

$$Z = \sum_{\text{states}} e^{-\beta H[\text{states}]}, \tag{1}$$

and related correlation functions of local observables. In the simplest case of classical statistical mechanics, where degrees of freedom commute (but also extendable to quantum statistical mechanics via the imaginary-time Feynman's path integral, which we will discuss shortly), the sum over states can often be reduced to an integral over the continuum degrees of freedom, such as a set of $\{p_i\}$ and $\{x_i\}$. Thus, in the context of statistical mechanics, harmonic (non-interacting) systems reduce to the problem of multiple (as we will see below sometimes infinite number of) Gaussian integrals over potentially coupled degrees of freedom, $\{q_i\}$. We thus pause to develop (hopefully review) the very important theoretical tool of Gaussian integrals calculus, that we will utilize over and over again here and throughout the course.

II. GAUSSIAN INTEGRALS

Given that the harmonic oscillator is a work-horse of theoretical physics, it is not suprising that Gaussian integrals are the key tool of theoretical physics. This is certainly clear in the computation of the partition function of a classical harmonic oscillator, as it involves Gaussian integration over fields **P** and **u**. However, as we will see, utilizing Feynman's path-integral formulation of quantum mechanics, Gaussian integrals are also central for computation in quantum statistical mechanics and more generally in quantum field theory.

A. one degree of freedom

Let us start out slowly with standard, scalar, one-dimension Gaussian integrals

$$Z_0(a) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}, \tag{2}$$

$$Z_1(a) = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{1}{2}ax^2} = -2\frac{\partial}{\partial a}Z_0(a) = \frac{1}{a}\sqrt{\frac{2\pi}{a}} = \frac{1}{a}Z_0, \tag{3}$$

$$Z_n(a) = \int_{-\infty}^{\infty} dx x^{2n} e^{-\frac{1}{2}ax^2} = \frac{(2n-1)!!}{a^n}Z_0, \tag{4}$$

that can be deduced from dimensional analysis, relation to the first basic integral $Z_0(a)$ (that can in turn be computed by a standard trick of squaring it and integrating in polar coordinates) or another generating function and Γ-functions

$$Z(a,h) = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+hx} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}a(x-h/a)^2} e^{\frac{1}{2}h^2/a} = e^{\frac{1}{2}h^2/a} Z_0(a), \tag{5}$$

$$= \sum_{n=0} \frac{h^{2n}}{(2n)!} Z_n(a). \tag{6}$$

Quite clearly, odd powers of x vanish by symmetry, and even powers give the $2n$-point correlation function,

$$C_{2n} \equiv \frac{Z_n(a)}{Z_0(a)} = \langle x^{2n} \rangle, \tag{7}$$

$$= \frac{1}{Z_0(a)} \frac{\partial^{2n}}{\partial h^{2n}} Z(a,h)|_{h=0} = (2n-1)!!(C_2)^n, \tag{8}$$

where $C_2 = G_0$ is a 2-point correlation function, that is also referred to as the propagator, G_0 of the harmonic theory (subscript 0 denotes the harmonic nature of the propagator).

A useful generalization of above Gaussian integral calculus is to integrals over complex numbers. Namely, from above we have

$$I_0(a) \; = \; \int_{-\infty}^{\infty} \frac{dxdy}{\pi} e^{-a(x^2+y^2)} = \frac{1}{a} = \int \frac{d\bar{z}dz}{2\pi i} e^{-a\bar{z}z}, \tag{9}$$

where in above we treat $\bar{z}, z$ as independent complex variables and the normalization is determined by the Jacobian of the transformation from x, y pair. This integral will be envaluable for path integral quantization and analysis of bosonic systems described by complex fields, $\overline{\psi}, \psi$, e.g., for statistical mechanics of superfluids and, more generally for the xy-model.

B. N degrees of freedom

This calculus can be straightforwardly generalized to multi-variable coupled Gaussian integrals characterized by an $N \times N$ symmetric matrix $(\mathbf{A})_{ij}$, as would appear in a Hamiltonian for N coupled harmonic oscillators,

$$Z_0(\mathbf{A}) \; = \; \int_{-\infty}^{\infty} [d\mathbf{x}] e^{-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}} = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{a_i}} = \sqrt{\frac{(2\pi)^N}{\det \mathbf{A}}}, \tag{10}$$

$$Z_1^{ij}(\mathbf{A}) \; = \; \int_{-\infty}^{\infty} [d\mathbf{x}] x_i x_j e^{-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}} = \mathbf{A}_{ij}^{-1} Z_0, \tag{11}$$

$$Z(\mathbf{A}, \mathbf{h}) \; = \; \int_{-\infty}^{\infty} [d\mathbf{x}] e^{-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} + \mathbf{h}^T \cdot \mathbf{x}} = e^{\frac{1}{2}\mathbf{h}^T \cdot \mathbf{A}^{-1} \cdot \mathbf{h}} Z_0, \tag{12}$$

computed by diagonalizing the symmetric matrix $\mathbf{A}$ and thereby decoupling the N-dimensional integral into a product of N independent scalar Gaussian integrals (4), each characterized by eigenvalue a_i, and then converting back into representation-independent form. In above we also defined a common multi-integral notation $\int_{-\infty}^{\infty} [d\mathbf{x}] \equiv \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dx_1 \ldots dx_N = \prod_i^N \left[\int_{-\infty}^{\infty} dx_i \right]$.

As a corollary of these Gaussian integral identities we have two more key results for a Gaussian random variable $\mathbf{x}$ (obeying Gaussian statistics), with variance $\mathbf{A}_{ij}^{-1}$,

$$Z[\mathbf{h}] \; \equiv \; \langle e^{\mathbf{h}^T \cdot \mathbf{x}} \rangle Z_0 = e^{\frac{1}{2}\langle (\mathbf{h}^T \cdot \mathbf{x})^2 \rangle} Z_0 = e^{\frac{1}{2}\mathbf{h}^T \cdot \mathbf{G} \cdot \mathbf{h}} Z_0, \tag{13}$$

$$\langle \mathbf{x}_i \mathbf{x}_j \rangle \; \equiv \; G_{ij}^0 = \frac{1}{Z_0} \int_{-\infty}^{\infty} [d\mathbf{x}] x_i x_j e^{-\frac{1}{2}\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}} = \mathbf{A}_{ij}^{-1}, \tag{14}$$

$$= \frac{1}{Z_0} \frac{\partial^2}{\partial h_i \partial h_j} Z[\mathbf{h}]|_{\mathbf{h}=0} = \frac{\partial^2}{\partial h_i \partial h_j} \ln Z[\mathbf{h}]|_{\mathbf{h}=0} \tag{15}$$

with the first identity the relative of the Wick's theorem, which will be extremely important for computation of Gaussian correlation functions. $Z[\mathbf{h}]$ is called the generating function for correlators of $\mathbf{x}$, because its n-th derivative with respect to h_i gives n-point correlation function of x_i.

As we will see in later, application of these identities to physical harmonic oscillator systems, they immediately reproduce the equipartition theorem ($\frac{1}{2}k_BT$ per classical quadratic degree of freedom), as in e.g., phonons in a solid.

C. Propagators and Wick's theorem for scalar field theory

Above multi-variable Gaussian calculus can now be straightforwardly generalized to functional Gaussian calculus, which will allow us to do statistical field theory. To this end we make the following identifications:

$$i \rightarrow \mathbf{x}, \tag{16}$$

$$x_i \rightarrow \phi(\mathbf{x}), \tag{17}$$

$$A_{ij} \rightarrow \Gamma(\mathbf{x}, \mathbf{x}'), \tag{18}$$

$$h_i \rightarrow h(\mathbf{x}), \tag{19}$$

namely, the discrete index i that labels the dynamical degree of freedom becomes a continuous label for a point in space, $\mathbf{x}$, the i-th dynamical variable x_i generalizes to a field $\phi(\mathbf{x})$ at a spatial point $\mathbf{x}$, the coupling matrix A_{ij} goes over to the continuous operator $\Gamma(\mathbf{x}, \mathbf{x}')$, and the external field h_i goes over to external field $h(\mathbf{x})$. With this, we can simply transcribe our earlier discrete Gaussian calculus to *functional* Gaussian calculus, keeping in mind that strictly speaking the latter is defined by the former through discretization of spatial field label $\mathbf{x}$.

For pedagogical clarity it is convenient to illustrate function integral calculus with a field theory of a real scalar field $\phi(\mathbf{x})$ (for a quantum dynamical fields, $\mathbf{x} = (\tau, \mathbf{r})$, with Euclidean imaginary time action $S[\phi(\mathbf{x})]$ replacing the classical Hamiltonian functional), governed by

$$H[\phi(\mathbf{x})] = \frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{x}'} \phi(\mathbf{x})\Gamma(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x}') - \int_{\mathbf{x}} h(\mathbf{x})\phi(\mathbf{x}), \tag{20}$$

with an external source field $h(\mathbf{x})$. Utilizing Gaussian integral calculus, the associated generating (partition) function is then given by

$$Z[h(\mathbf{x})] = \int \mathcal{D}\phi(\mathbf{x}) e^{-\frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{x}'}\phi(\mathbf{x})\Gamma(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}') + \int_{\mathbf{x}} h(\mathbf{x})\phi(\mathbf{x})}, \tag{21}$$

$$= e^{\frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{x}'} h(\mathbf{x})\Gamma^{-1}(\mathbf{x},\mathbf{x}')h(\mathbf{x}')}, \tag{22}$$

where the $\Gamma^{-1}(\mathbf{x},\mathbf{x}')$ is an inverse of $\Gamma(\mathbf{x},\mathbf{x}')$, and $\int\mathcal{D}\phi(x)\equiv\int[d\phi(\mathbf{x})]\equiv\prod_{\mathbf{x}}\left[\int_{-\infty}^{+\infty}d\phi(\mathbf{x})\right]$. For a translationally invariant case $\Gamma(\mathbf{x}-\mathbf{x}')$, inverse is computed by a Fourier transformation, namely,

$$\Gamma^{-1}(\mathbf{x}-\mathbf{x}')=\int_{\mathbf{k}}\frac{1}{\tilde{\Gamma}(\mathbf{k})}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \tag{23}$$

Using $Z[h(\mathbf{x})]$ the correlators are straightforwardly computed by simply differentiating with respect to $h(\mathbf{x})$,

$$G(\mathbf{x},\mathbf{x}')=\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle=\frac{1}{Z}\frac{\delta^2 Z[h(\mathbf{x})]}{\delta h(\mathbf{x})\delta h(\mathbf{x}')}\bigg|_{h=0}=\Gamma^{-1}(\mathbf{x},\mathbf{x}'). \tag{24}$$

The "connected" correlation functions

$$\begin{aligned}
G_c(\mathbf{x},\mathbf{x}') &= \langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_c\equiv\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle-\langle\phi(\mathbf{x})\rangle\langle\phi(\mathbf{x}')\rangle && (25)\\
&= \langle[\phi(\mathbf{x})-\langle\phi(\mathbf{x})\rangle]\,[\phi(\mathbf{x}')-\langle\phi(\mathbf{x}')\rangle]\rangle, && (26)\\
&= \frac{\delta^2\ln Z[h(\mathbf{x})]}{\delta h(\mathbf{x})\delta h(\mathbf{x}')}\bigg|_{h=0}\equiv\frac{\delta^2 W[h(\mathbf{x})]}{\delta h(\mathbf{x})\delta h(\mathbf{x}')}\bigg|_{h=0}, && (27)
\end{aligned}$$

where $W[h(\mathbf{x})]=\ln Z[h(\mathbf{x})]$ is a generating function for *connected* correlation functions, with disconnected parts cancelled by the differentiation of the normalization $1/Z[h(\mathbf{x})]$.

Using $Z[h(\mathbf{x})]$ above we immediately obtain the powerful Wick's theorem valid for Gaussian fields only (i.e., those characterized by a harmonic [quadratic, non-interacting] Hamiltonian). Namely,

$$\begin{aligned}
\langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\ldots\phi(\mathbf{x}_{2n})\rangle &= \frac{1}{Z}\frac{\delta^{2n}Z[h(\mathbf{x})]}{\delta h(\mathbf{x}_1)\delta h(\mathbf{x}_2)\delta h(\mathbf{x}_3)\ldots\delta h(\mathbf{x}_{2n})}\bigg|_{h=0},\\
&= G(\mathbf{x}_1,\mathbf{x}_2)G(\mathbf{x}_3,\mathbf{x}_4)\ldots G(\mathbf{x}_{2n-1},\mathbf{x}_{2n})\\
&\quad+\text{all other pairings of }\mathbf{x}_i,\mathbf{x}_j, && (28)
\end{aligned}$$

and vanishing for correlators odd number of fields.

Above Wick's theorem directly applies to a *classical* statistical field theory of commuting fields. Thanks to a path-integral formulation of a quantum field theory (that maps it onto an effective $d+1$-dimensional commuting, classical statistical field theory)[10, 18], with a slight modification, the theorem also extends to a quantum field theory for time-ordered correlation functions in a ground state $|0\rangle$,

$$\begin{aligned}
\langle 0|T_\tau\left(\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\ldots\phi(\mathbf{x}_{2n})\right)|0\rangle &= G(\mathbf{x}_1,\mathbf{x}_2)G(\mathbf{x}_3,\mathbf{x}_4)\ldots G(\mathbf{x}_{2n-1},\mathbf{x}_{2n})\\
&\quad+\text{all other pairings of }\mathbf{x}_i,\mathbf{x}_j, && (29)
\end{aligned}$$

A more general form of the quantum Wick's theorem at the level of operators is given by

$$\begin{aligned}
T_\tau\left(\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\ldots\phi(\mathbf{x}_n)\right) &= :\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)\phi(\mathbf{x}_5)\ldots\phi(\mathbf{x}_n): \\
&= \overbrace{\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)}:\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)\phi(\mathbf{x}_5)\ldots\phi(\mathbf{x}_n): \\
&\quad +\text{all other } \textit{single} \text{ pair } (\mathbf{x}_i,\mathbf{x}_j) \text{ contraction} \\
&= \overbrace{\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)}\,\overbrace{\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)}:\phi(\mathbf{x}_5)\ldots\phi(\mathbf{x}_n): \\
&\quad +\text{all other } \textit{double} \text{ pair } (\mathbf{x}_i,\mathbf{x}_j),\ (\mathbf{x}_k,\mathbf{x}_l) \text{ contraction} \\
&= \overbrace{\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)}\ldots\overbrace{\phi(\mathbf{x}_{n-1})\phi(\mathbf{x}_n)} \\
&\quad +\text{all other } n/2 \text{ pairs, if } n \text{ is even} \\
&= \overbrace{\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)}\ldots\overbrace{\phi(\mathbf{x}_{n-2})\phi(\mathbf{x}_{n-1})}\,\phi(\mathbf{x}_n) \\
&\quad +\text{all other } (n-1)/2 \text{ pairs, if } n \text{ is odd,}
\end{aligned} \tag{30}$$

where the contraction of a pair of fields is defined to be

$$\overbrace{\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)} \equiv T_\tau\left(\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\right) - :\phi(\mathbf{x}_1)\phi(\mathbf{x}_2):, \tag{31}$$

$:\hat{O}:$ is the normal ordered arrangments of operators with creation operators to the left of annihilation operators. Evaluation of the expectation value in the vacuum gives the path-integral expression, (29).

D. Generating functions

As we have seen above in Eq.(28), $Z[h(\mathbf{x})]$ is a generating function for n-point correlation functions, that appear as functional coefficients in the expansion of $Z[h(\mathbf{x})]$ in powers of $h(\mathbf{x})$. There are a number of other generating functions that importantly appear in field theory and I briefly summarize their properties and relationship. For further details, I direct the reader to the wonderful textbook by Zinn Justin[10] and other field theory books, e.g., one by Lewis Ryder.

1. $Z(h)$ full generating function of **all** *diagrams*

As I discussed above n-th derivative of this $Z(h)$ generating function generates n-point correlation functions:

$$\langle\phi_1\phi_2\ldots\phi_n\rangle_0 = \frac{\partial^n}{\partial h^n}\Big|_{h=0} Z(h).$$

It contains contributions from all diagrammatic ways to construct an n-point correlation function.

2. $W(h) = \ln Z(h)$ *- generating function for* **connected** *diagrams*

However, a generating function that generates *all* contributions to a correlator is not exactly convenient and in fact carries unnecessary information. Cearly, once connected components of a diagram are generated, the disconnected components are constructed as various powers of connected ones. Thus, it is more convenient and economical to focus only on the connected diagrams and therefore to work with a corresponding generating function of connected-only correlators. Although it takes a bit of thought to prove in full generality[10] (I encourage you to demonstrate this explicitly for a few simple cases, as I will do in the next section), the process of taking the logarithm of $Z(h)$ eliminates all the disconnected diagrams (those that fall apart into multiple pieces). As the simplest illustrative example, let's look at the 2-point correlation function, generated by $W(h)$, where subscript c stands for "connected",

$$\begin{aligned}
\langle\phi_1\phi_2\rangle_c &= \frac{\partial^2 W}{\partial h^2}\Big|_{h=0} = \frac{\partial^2}{\partial h^2}\Big|_{h=0} \ln Z(h) = \frac{\partial}{\partial h}\Big|_{h=0}\left(\frac{1}{Z}\frac{\partial}{\partial h}Z\right), && (32)\\
&= -\left(\frac{\partial Z}{\partial h}\right)^2_{h=0} + \frac{1}{Z}\frac{\partial^2}{\partial h^2}\Big|_{h=0} Z(h), && (33)\\
&= \langle\phi_1\phi_2\rangle - \langle\phi_1\rangle\langle\phi_2\rangle = \langle(\phi_1 - \langle\phi_1\rangle)\,(\phi_2 - \langle\phi_1\rangle)\rangle. && (34)
\end{aligned}$$

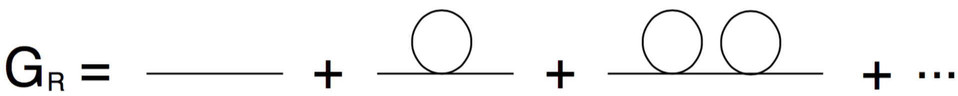
Thus $W(h)$ generates a 2-point function, where disconnected pieces $\langle\phi_1\rangle\langle\phi_2\rangle$ have been subtracted out, i.e., do not appear. This is true for arbitrary n-point correlation function generated by $W(h)$.

3. $\Gamma(\varphi) = \varphi h - W(h)$ *- generating function for* **one-particle irreducible 1PI** *diagrams diagrams*

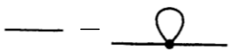
It can be shown that even $W(h)$ contains too much information, in that even all the connected diagrams can be generated from a more "economical" ("powerful") generating function $\Gamma(\varphi)$, that is the Legendre transform of $W(h)$, with $\varphi \equiv \frac{\partial W(h)}{\partial h}$, allowing one to eliminate the external (e.g., magnetic) field, h in favor of the background order parameter field (magnetization) φ. This is in close analogy with the way that a Hamiltonian $H(p)$ as a function of momentum p is a Legendre transform of the Lagrangian $L(\dot{q})$, trading $\dot{q}$ dependence for momentum $p = \partial L/\partial \dot{q}$. The key point about

$$\Gamma(\varphi) = \sum_{n=0} \frac{1}{n!}\Gamma^{(n)}\varphi^n,$$

that is again not obvious to demonstrate, is that its Taylor expansion generates 1PI diagrams, $\Gamma^{(n)}$ with n external "legs amputated". 1PI diagrams give vertex functions, that cannot be cut into disconnected graphs by cutting a single line. In the simplest case of a 2-point correlation function, it is easy to check that while $W(h)$ generates all corresponding connected diagrams, $\Gamma(\varphi)$ generates all 1PI diagrams to the inverse propagator $\Gamma^{(2)}$, which, when inverted recovers all the "missing" non-1PI diagrams contributing to the 2-point correlation function (propagator). Specifically, to lowest one-loop order, the *connected* 2-point function (the Greens function) $W^{(2)}$, generated by $W(h)$ is diagrammatically given by a geometric series of infinite number of terms,



In contrast, the one-loop order 1PI 2-point correlation function $\Gamma^{(2)}$, generated by $\Gamma(h)$ is simply given by just one correction term,

$\Gamma^{(2)}(\mathbf{k}, \mathbf{k}') \simeq$  (35)

with the geometric series generated from this one self-energy correction, when the propagator G is constructed from $\Gamma^{(2)}$ by inversion, i.e., $G \equiv W^{(2)} = 1/\Gamma^{(2)}$. This is discussed in more detail in problem 4 of homework set 2.

III. STATES AND THEIR COUNTING

Throughout the lectures we will go back and forth between discrete and continuum description of the degrees of freedom. In all calculations, even when done in the continuum limit, it is quite important to keep in mind the discrete and therefore finite nature of the degrees of freedom, with the continuum description being simply an efficient pnemonic for the underlying lattice model. This is always the case in any physical system and any pretense otherwise is misguided. This guarantees that no true short-scale (ultra-violet, UV) divergences actually ever arise, cutoff by the physical lattice structure always present in any physica matter system.

A. Discrete vs continuum description

Given that a volume of a unit cell is v and reciprocal space is quantized in units of $2\pi/L$, the relations between sums and integrals in real and reciprocal spaces are given by

$$\sum_{\mathbf{x}} \ldots = \frac{1}{v}\int d^d x \ldots, \tag{36}$$

$$\sum_{\mathbf{k}} \ldots = L^d \int \frac{d^d k}{(2\pi)^d} \ldots. \tag{37}$$

Also, we note the relation between the Kronecker δ and δ-function identities,

$$\sum_{\mathbf{x}}^{N} e^{i\mathbf{k}\cdot\mathbf{x}} = N\delta_{\mathbf{k},0}, \tag{38}$$

$$\sum_{\mathbf{x}}^{N} v e^{i\mathbf{k}\cdot\mathbf{x}} = vN\delta_{\mathbf{k},0}, \tag{39}$$

$$\int d^d x e^{i\mathbf{k}\cdot\mathbf{x}} = V\delta_{\mathbf{k},0} = \frac{(2\pi)^d}{(2\pi/L)^d}\delta_{\mathbf{k},0} = (2\pi)^d \delta^d(\mathbf{k}), \tag{40}$$

where $V = vN$.

B. Density of states

There will be many instances where our result is represented as a sum over the normal eigenmodes $\mathbf{k}$. If the summand is only a function the normal-mode frequency $\omega_{\mathbf{k}}$ (as will often be the case) it is convenient to replace the sum over $\mathbf{k}$ by an integral over ω, with the Jacobian of this transformation being the density of states $g(\omega)$, defined according to:

$$F = \sum_{\mathbf{k}} f(\omega_{\mathbf{k}}) = \int d\omega \left(\sum_{\mathbf{k}} \delta(\omega - \omega_{\mathbf{k}})\right) f(\omega), \tag{41}$$

$$= \int d\omega g(\omega) f(\omega), \tag{42}$$

where the number of states $\mathbf{k}$ per interval $d\omega$ around ω is given by the density of states

$$g(\omega) = \sum_{\mathbf{k}} \delta(\omega - \omega_{\mathbf{k}}) = L^d \int \frac{d^d k}{(2\pi)^d} \delta(\omega - \omega_{\mathbf{k}}), \tag{43}$$

where I ignored the distinct polarization modes (number of components of the field, taking it to be one). I note that sometimes $g(\omega)$ is defined without the volume factor L^d, corresponding to the density of states per unit of volume. Also, by construction, $g(\omega)$ satisfies the sum rule $\int d\omega g(\omega) = N$.

The limit on large k is given by G set by the first BZ, corresponding to uv cutoff by the lattice spacing in $\mathbf{R}_n$. There is also infrared cutoff set by the system size, L or equivalently in momentum space by discreteness of $k = \frac{2\pi}{L} p$.

There are two canonical models of phonons, the Debye model with $\omega_k = ck$ and the Einstein model with $\omega_k = \omega_0$. The density of states for these "toy" models are straightforwardly computed to be

$$g_{Debye}(\omega) = L^d \int \frac{d^d k}{(2\pi)^d} \delta(\omega - ck), \tag{44}$$

$$= L^d \frac{S_d}{(2\pi)^d c^d} \, \omega^{d-1}, \quad \text{for } 0 < \omega < \omega_{Debye} \tag{45}$$

$$g_{Einstein}(\omega) = L^d \int \frac{d^d k}{(2\pi)^d} \delta(\omega - \omega_o), \tag{46}$$

$$= N \delta(\omega - \omega_o), \tag{47}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is a surface area of a d-dimensional sphere and ω_D is defined by $N = \int_0^{\omega_D} \omega g_{Debye}(\omega)$.

IV. CLASSICAL STATISTICAL FIELD THEORY

For completeness we go back to the fundamental goal of statistical mechanics, namely to calculate the partition function, Eq. (1), that, as we have seen in the magnetism lecture for an Ising lattice ferromagnet,

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2}\beta \sum_{ij} J_{ij} \sigma_i \sigma_j}, \tag{48}$$

$$= Z_{J0}^{-1} \sum_{\{\sigma_i\}} \int \mathcal{D}\phi_i e^{-\frac{1}{2}\beta^{-1} \sum_{ij} J_{ij}^{-1} \phi_i \phi_j + \sum_i \sigma_i \phi_i}, \tag{49}$$

$$= Z_{J0}^{-1} \int \mathcal{D}\phi_i e^{-\frac{1}{2}\beta^{-1} \sum_{ij} J_{ij}^{-1} \phi_i \phi_j + \sum_i \ln\cosh\phi_i} \equiv \int \mathcal{D}\phi_i e^{-\beta H_{\rm eff}(\{\phi_i\})}. \tag{50}$$

can be transformed into a ϕ^4 field theory, with the first step often referred to as the Hubbard-Stratonovich (HS) transformation[4, 7, 10], but is nothing more than a Gaussian integral (run in reverse) of the previous sections. Going to the continuum, $\phi_i \to \phi(\mathbf{x})$, we obtain a field theoretic expression for the Ising model partition function, that near PM-FM critical point can we captured by a ϕ^4 field theory,

$$Z = \int \mathcal{D}\phi(\mathbf{x}) e^{-\beta H_{\text{eff}}[\phi(\mathbf{x})]}, \tag{51}$$

with the effective Hamiltonian given by (dropping an irrelevant additive constant $k_B T \ln Z_{J0}$ and unimportant higher order ϕ_i terms)

$$H_{\text{eff}}[\phi(\mathbf{x})] = \int_{\mathbf{x}} \left[\frac{1}{2} K (\boldsymbol{\nabla}\phi)^2 + \frac{1}{2} t \phi^2 + \frac{1}{4} u \phi^4 \right], \tag{52}$$

and the effective coupling constants

$$K = \frac{(k_B T)^2 a^{2-d}}{J_0}, \tag{53}$$

$$t = k_B T a^{-d} \left(\frac{k_B T}{J_0} - 1 \right), \tag{54}$$

$$u = k_B T a^{-d}/3. \tag{55}$$

Because of its generic nature, this model prominently appears in condensed matter and particle field theory studies, and in the context of critical phenomena is referred to as the Landau-Ginzburg Hamiltonian for the coarse-grained continuum fields $\phi(\mathbf{x})$.

A. Challenges to exact solution

There are two challenging aspects of performing above functional integral over field $\phi(\mathbf{x})$ to compute the partition function. One is the first gradient (elastic) term and the other the nonlinear nature of the functional $H_{\text{eff}}[\phi(\mathbf{x})]$, arising from the last quartic term. In the absence of either of these obsticles the partition function is computable exactly. Let us deal with each of these challenges below.

1. *Gradient terms*

In the absence of a gradient term, values of $\phi(\mathbf{x})$ at each site $\mathbf{x}$ are independent, and the functional reduces to a product over $\mathbf{x}$ of independent ordinary integrals, one over each variable $\phi(\mathbf{x})$ labelled by discrete positions $\mathbf{x}$. As a warm up to a full treatment, we will analyze a model like this below.

However, as can be seen by discretization, $\int_{\mathbf{x}}(\boldsymbol{\nabla}\phi)^2 \simeq \sum_{\mathbf{x},\boldsymbol{\delta}}(\phi_{\mathbf{x}} - \phi_{\mathbf{x}+\boldsymbol{\delta}})^2$, the gradient term couples degrees of freedom at neighboring sites, precluding a direct independent integration over them. More generally, such coupling term can be written as

$$H_0[\phi(\mathbf{x})] = \frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{x}'}\phi(\mathbf{x})\Gamma(\mathbf{x},\mathbf{x}')\phi(\mathbf{x}'), \tag{56}$$

and can be decoupled by transforming to normal modes of the coupling "matrix", $\Gamma(\mathbf{x},\mathbf{x}')$. For a translationally invariant case $\Gamma(\mathbf{x}-\mathbf{x}')$, the normal modes are simply the Fourier transform fields, $\tilde{\phi}(\mathbf{k})$,

$$\tilde{\phi}(\mathbf{k}) = \int_{\mathbf{x}}\phi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{57}$$

$$\phi(\mathbf{x}) = \int\frac{d^dk}{(2\pi)^d}\tilde{\phi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{58}$$

in terms of which any translationally invariant spatial coupling decouples in Fourier space,

$$H_0 = \frac{1}{2}\int_{\mathbf{x}}\int_{\mathbf{x}'}\phi(\mathbf{x})\Gamma(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}'), \tag{59}$$

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\tilde{\phi}(-\mathbf{k})\tilde{\Gamma}(\mathbf{k})\tilde{\phi}(\mathbf{k}) \equiv \frac{1}{2}\int_{\mathbf{k}}\phi_{-\mathbf{k}}\Gamma_{\mathbf{k}}\phi_{\mathbf{k}}, \tag{60}$$

$$= \frac{1}{2}\int_{\mathbf{k}}\tilde{\Gamma}(\mathbf{k})|\tilde{\phi}(\mathbf{k})|^2 = \frac{1}{2}\int_{\mathbf{k}}\tilde{\Gamma}(\mathbf{k})\left(\tilde{\phi}_r^2(\mathbf{k}) + \tilde{\phi}_i^2(\mathbf{k})\right)^2. \tag{61}$$

Above, I defined $\int_{\mathbf{k}} \equiv \int\frac{d^dk}{(2\pi)^d}$ and in a slight abuse of notation, dropped tilde, using the same symbol for two distinct functions, $\phi(\mathbf{x})$ and its Fourier transform $\tilde{\phi}(\mathbf{k}) \equiv \phi_{\mathbf{k}}$. I trust that no confusion should arise as I distinguish these by their arguments, and will use $\mathbf{k}$ as a subscript.

I further note, that, in fact a Fourier transformation above does not quite decouple the modes, retaining coupling $\phi(\mathbf{k})$ to its single partner $\phi_{-\mathbf{k}}$. However, the remaining coupling is just a 2×2 matrix for each $\mathbf{k}$ and is easily decoupled by a sum and differences ($\pi/4$ "rotation"). Furthermore, for a real field $\phi(\mathbf{x})$, it is easy to verify that its Fourier transform is constrained by a condition $\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^*$ and so this final decoupling is into real and imaginary parts of the field $\phi_{\mathbf{k}}$, as indicated in the last line above.

Specializing to the gradient, as the case of most common interest, and transitioning to Fourier modes, we find that $\tilde{\Gamma}(\mathbf{k}) \equiv \Gamma_{\mathbf{k}} = k^2$, i.e.,

$$H_0 = \frac{1}{2}\int_{\mathbf{x}}(\boldsymbol{\nabla}\phi)^2 = \frac{1}{2}\int_{\mathbf{k}}k^2|\phi_{\mathbf{k}}|^2. \tag{62}$$

2. *Nonlinearities, interactions, mode coupling*

The other, much more serious obsticle to a direct functional integral computation of the partition function is the nonlinear nature of the effective Hamiltonian, namely the quartic interaction $H_{\text{int}} = u\int_{\mathbf{x}} \phi^4$. For simplicity of notation I redefined u such that no factor $\frac{1}{4}$ appears. Also, because our focus is on the critical point, $k_B T \approx k_B T_c$, I will also set $k_B T = 1$. I note that it in itself, this nonlinearity would not be a big problem in the absence of spatial coupling discussed above, as the partition function would amount to a product over $\mathbf{x}$ of independent ordinary quartic integrals. For example, for $K = 0$ it would give a very simple and exact result,

$$Z = \prod_{\mathbf{x}} \left[\int d\phi_{\mathbf{x}} e^{-\frac{1}{2}t\phi_{\mathbf{x}}^2 - u\phi_{\mathbf{x}}^4}\right] = \left[\int d\phi e^{-\frac{1}{2}t\phi^2 - u\phi^4}\right]^N, \tag{63}$$

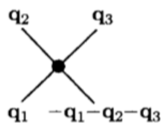
$$= \left[(t/8u)^{1/2} e^{\frac{t^2}{32u}} K_{\frac{1}{4}}\left(t^2/32u\right)\right]^N, \tag{64}$$

where $K_{\frac{1}{4}}[x]$ is a modified Bessel K function of order 1/4.

In the presence of both the gradient terms and the nonlinearities, in terms of the Fourier modes $\phi_{\mathbf{q}}$ (necessary to decouple the gradient term), the quartic nonlinearity couples different $\mathbf{q}$ models,

$$H = H_0 + H_{\text{int}} \tag{65}$$

$$= \frac{1}{2}\int_{\mathbf{q}} (Kq^2 + t)|\phi_{\mathbf{q}}|^2 + u\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} \phi_{\mathbf{q}_1}\phi_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\phi_{\mathbf{q}_4}(2\pi)^d\delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4), \tag{66}$$

$= H_0 +$ 

, (67)

where the quartic vertex graphically represents u coupling, with translational invariance enforcing the vanishing of the sum of the four momenta.

B. Noninteracting $u = 0$ analysis

We first consider a harmonic theory, i.e., in the absence of nonlinearities, taking $u = 0$. We note that for stablity of the theory this limit is only possible for $t > 0$ (above T_c). In this case, a statistical analysis reduces to Gaussian functional integrals of decoupled modes $\mathbf{q}$, allowing us to take advantage of the calculus that we developed in the previous section.

The harmonic partition function can be easily obtained using (10),

$$Z_0 = \int \mathcal{D}\phi(\mathbf{r}) e^{-\frac{1}{2}\int_{\mathbf{r}}\left[K(\boldsymbol{\nabla}\phi)^2 + t\phi^2\right]}, \tag{68}$$

$$= \int \mathcal{D}\phi_{\mathbf{q}} e^{-\frac{1}{2}\int_{\mathbf{q}}(Kq^2+t)|\phi_{\mathbf{q}}|^2}, \tag{69}$$

$$= \prod_{\mathbf{q}} \left[\frac{2\pi}{Kq^2+t}\right], \tag{70}$$

where for every $\mathbf{q}$ there are two, real and imaginary modes (hence no square-root). Similarly the two-point correlation function is also straightforwardly obtained using (15), (23) and (24),

$$G_0(\mathbf{r}-\mathbf{r}') = \langle\phi(\mathbf{r})\phi(\mathbf{r}')\rangle_0 = \Gamma^{-1}(\mathbf{r}-\mathbf{r}'), \tag{71}$$

$$= \int_{\mathbf{q}} \frac{1}{Kq^2+t} e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}, \tag{72}$$

$$= \frac{1}{K}\frac{e^{-|\mathbf{r}-\mathbf{r}'|/\xi_+}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \mathrm{d}=3, \tag{73}$$

where the second equality (line) can also most simply be obtained by the equipartition theorem, and the high temperature correlation length is given by

$$\xi_+ = \sqrt{\frac{K}{t}} \sim t^{-\nu},\ \nu = \frac{1}{2}\ \text{, in harmonic approximation.} \tag{74}$$

The exponential fall-off of the real-space correlation function (propagator) captures short-range correlations characterizing the high temperature disordered phase, where fields $\phi(\mathbf{r})$ at distant points fluctuate in uncorrelated way, so at large separations the correlator averages to zero.

I also note that in Fourier space, we have,

$$\tilde{G}_0(\mathbf{q},\mathbf{q}') = \langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle_0, \tag{75}$$

$$= \frac{1}{Kq^2+t}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'), \tag{76}$$

$$\equiv \tilde{G}_0(\mathbf{q})(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}'), \tag{77}$$

with the δ-function enforcing momentum conservation in this translationally-invariant model.

Because of the quadratic nature of H for $u = 0$, all physical observables can be straightforwardly calculated using Gaussian functional integral calculus of previous sections. For example, the uniform linear susceptibility χ of the magnetization $M = \int d^d r \phi(\mathbf{r})$ in response to a uniform external field h (included in the Hamiltonian via $H_h = -h \int_{\mathbf{r}} \phi(\mathbf{r}) = -h\langle M \rangle$) is given by

$$\langle M(h \to 0) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \left[\int d^d r_1 \phi(\mathbf{r}_1) \right] e^{-\beta H_{h=0}[\phi] + \beta h \int_{\mathbf{r}} \phi(\mathbf{r})}, \tag{78}$$

Thus, linear susceptibility, defined by $\langle M \rangle =_{h\to 0} \chi_{\text{unif}}\ h$, is given by

$$\chi_{\text{unif}} = \frac{1}{k_B T} \langle MM \rangle = \frac{1}{k_B T} \int d^d r_1 d^d r_2 \langle \phi(\mathbf{r}_1) \phi(\mathbf{r}_2) \rangle, \tag{79}$$

$$= \frac{1}{k_B T} \int d^d R \int d^d (r_1 - r_2) G(|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{V}{k_B T} G(\mathbf{q} = 0), \tag{80}$$

$$= \frac{L^d}{t} \sim t^{-\gamma}, \quad \gamma = 1, \quad \text{in harmonic approximation.} \tag{81}$$

The above relation of the susceptibility to a correlation function $\chi = \frac{1}{k_B T} \langle MM \rangle$ is an example of a general and important Fluctuation-Dissipation theorem.

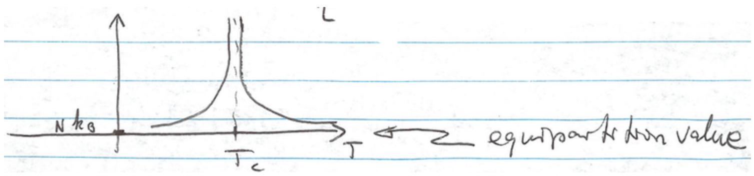


FIG. 1: Uniform magnetic susceptibility χ_{unif} in a harmonic approximation near a critical point, T_c.

Heat capacity (focussing on its singular, critical part only) can be similarly calculated for $t > 0$,

$$C_{\text{sing}} = -T \frac{\partial^2 F}{\partial T^2} \approx -T_c \frac{\partial^2 F}{\partial t^2} \approx T_c \frac{\partial E}{\partial t}, \tag{82}$$

$$= -\frac{\partial}{\partial t} \int d^d r_1 \langle \phi^2(\mathbf{r}_1) \rangle \approx \int d^d r_1 d^d r_2 \langle \phi^2(\mathbf{r}_1) \phi^2(\mathbf{r}_2) \rangle, \tag{83}$$

$$= V \int \frac{d^d q}{(2\pi)^d} \frac{1}{(Kq^2 + t)^2} \approx \frac{V}{K^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + \xi^{-2})^2} \sim \frac{V}{K^2} \begin{cases} a^{4-d}, & \text{for } d > 4, \\ \xi_+^{4-d}, & \text{for } d < 4. \end{cases} \tag{84}$$

giving $C_{\text{sing}} \sim t^{-\alpha}$, with $\alpha = \alpha_{MF} = 0$ for $d > 4$ and $\alpha = (4 - d)/2$ for $d < 4$ within a harmonic approximation. This reflects an important observation, that for $d > 4$ MFT remains valid, but breaks down for $d < 4$ due to importance of thermal fluctuations.

C. interacting $u > 0$ analysis

We now return to the full nonlinear theory, that does not admit an exact analysis. One way to make progress is to consider a limit of small nonlinear coupling u and to perform a perturbative expansion in u. The precise criterion for the validity of this approach can be determined a posteriori, by noting a value of u at which perturbation theory fails to converge. Perhaps not surprisingly, this will be given by the Ginzburg criterion from Lecture 3.

1. *Perturbative expansion for a single-mode "toy" ϕ^4 model*

To warm up for a full treatment, we first consider a single-mode "toy" ϕ^4 theory characterized by

$$H = \frac{1}{2}t\phi^2 + u\phi^4, \tag{85}$$

corresponding to $K = 0$ and/or taking only a single momentum mode $\mathbf{q}$. The additional advantage is that the analysis of the model reduces to a conventional integral and thus admits an exact solution as found (64).

We first recall Gaussian integrals that will be necessary for the perturbative expansion in $u\phi^4$,

$$\int_{-\infty}^{\infty} d\phi\ \phi^n e^{-\frac{1}{2}t\phi^2} = \frac{\partial^n}{\partial h^n}\,|_{h=0}\ Z(h) = \begin{cases} Z_0(n-1)!!\left(\frac{1}{t}\right)^{n/2}, & \text{for n even,} \\ 0, & \text{for n odd.} \end{cases} \tag{86}$$

$$\tag{87}$$

with

$$Z(h) \equiv \int_{-\infty}^{\infty} d\phi e^{-\frac{1}{2}t\phi^2 + h\phi} = Z(0)e^{\frac{1}{2}h^2/t}, \tag{88}$$

and $Z(0) = Z_0 = \sqrt{2\pi/t}$. Above is equivalent to a zero-dimensional Wick's theorem,

$$\langle\phi^{2n}\rangle = (2n-1)!!\langle\phi^2\rangle^n = (2n-1)!!G_0^n, \tag{89}$$

where the noninteracting ($u = 0$) propagator in this "toy" model is given by

$$G_0 = \langle\phi\phi\rangle_0 = \frac{\int_{-\infty}^{\infty} d\phi\ \phi^2 e^{-\frac{1}{2}t\phi^2}}{\int_{-\infty}^{\infty} d\phi\ e^{-\frac{1}{2}t\phi^2}} = \frac{1}{t}. \tag{90}$$

The $(2n-1)!!$ can be straightforwardly demonstrated in a number of ways. One is by Taylor-expanding the generating function $Z(h)$ in powers of $h^2/2t$ and comparing term by term to the series expansion in h in (87). Alternatively, it can be obtained diagrammatically, by thinking of ϕ^{2n} a 2n-point vertex and counting the total distinct ways to "contract" (connect up into a loop) all n pairs of vertex legs.

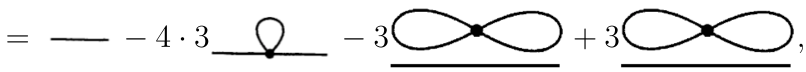
Wick's theorem thereby allows a computation of any average of arbitrary power of ϕ in terms of a corresponding power of the pair-correlator of the Gaussian ($u = 0$) theory. Such perturbative expansion leads to a growing number of terms. It is convenient to keep track of them in terms of pictorial Feynman diagrams named after their inventor[7, 10]. These diagrams are nothing more than graphical representaton of all pairings of $2n$ fields in the perturbative expansion.

The most elementary diagram is the harmonic propagator,

$$G_0 \;=\; \langle\phi^2\rangle_0 = \frac{\partial^2}{\partial h^2}|_{h=0} e^{\frac{1}{2}\frac{h^2}{t}} = \frac{\partial^2}{\partial h^2}|_{h=0} \bullet\!\!-\!\!\bullet \;=\; \text{———} = \frac{1}{t}. \tag{91}$$

The propagator G in the full nonlinear theory can now be calculated perturbatively in quartic interaction u,

$$\begin{aligned}
G = \langle\phi^2\rangle \;&=\; \frac{\int d\phi\phi^2 e^{-\frac{1}{2}t\phi^2 - u\phi^4}}{\int d\phi e^{-\frac{1}{2}t\phi^2 - u\phi^4}}, && (92)\\
&= \frac{\int d\phi\; \phi^2[1 - u\phi^4 + \ldots]e^{-\frac{t}{2}\phi^2}}{\int d\phi[1 - u\phi^4 + \ldots]e^{-\frac{t}{2}\phi^2}}, && (93)\\
&= \frac{\langle\phi^2\rangle_0 - u\langle\phi^6\rangle_0 + \ldots}{1 - u\langle\phi^4\rangle_0 + \ldots} = \langle\phi^2\rangle_0 - u\langle\phi^6\rangle_0 + u\langle\phi^2\rangle_0\langle\phi^4\rangle_0 + \ldots, && (94)\\
&= G_0 - u5!!G_0^3 + u3!!G_0^3 + \ldots, && (95)
\end{aligned}$$



(96)

$$\simeq \frac{1}{t} - 12\frac{1}{t}\frac{u}{t}\frac{1}{t} = \frac{1}{t}\big(1 - \frac{12u}{t^2}\big), \tag{97}$$

where the 3rd term arises from the correction to the denominator, the harmonic partition function, Z_0 and cancels the "disconnected" part of the second term $\langle\phi^6\rangle_0$. Rewriting this result as the correction to the inverse propagator,

$$\Gamma^{(2)} \equiv G^{-1} \simeq G_0^{-1}\left(1 + 12u/t^2\right) = G_0^{-1} + 12uG_0, \tag{98}$$

gives the correction to the reduced temperature t and therefore a reduction in the transition temperature, T_c,

$$\begin{aligned}
t_R \;&=\; a(T - T_c^R) \approx t + 12u/t, && (99)\\
&\Rightarrow \delta T_c \approx -12u/t. && (100)
\end{aligned}$$

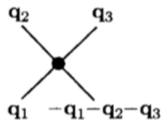
I note that a correction to the inverse propagator, which is a one-particle-irreducible (1PI) function, $\Gamma^{(2)}$[10], simply comes from a single-loop diagram (96), and does not involve disconnected nor non-1PI diagrams. This is a general rule for 1PI correlation functions.[10]

2. Perturbative expansion for the full multi-mode ϕ^4 model

Having warmed up on a single mode model, we now repeat the analysis for the full scalar ϕ^4 (Ising model) in d-dimensions

$$H = H_0 + H_{\text{int}} \tag{101}$$

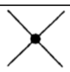
$$= \frac{1}{2}\int_{\mathbf{q}} (Kq^2 + t)|\phi_{\mathbf{q}}|^2 + u\int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4} \phi_{\mathbf{q}_1}\phi_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\phi_{\mathbf{q}_4}(2\pi)^d\delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4), \tag{102}$$

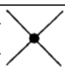
$= H_0 +$  , (103)

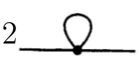
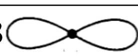
where the quartic vertex graphically represents u nonlinearity and couples different momentum modes. The harmonic part is characterized by free propagator,

$$\langle\phi(\mathbf{k})\phi(\mathbf{k}')\rangle_0 = (2\pi)^d\delta^d(\mathbf{k} + \mathbf{k}')\frac{1}{Kk^2 + t} \equiv (2\pi)^d\delta^d(\mathbf{k} + \mathbf{k}')G_0(k), \tag{104}$$

We now consider a calculation of the two-point correlation function (propagator) in the fully interacting ϕ^4 field theory by perturbatively expanding in the coupling u. To this end, mirroring the single mode analysis of the previous subsection, we find,

$\langle\phi(\mathbf{k})\phi(\mathbf{k}')\rangle = \dfrac{\int[d\phi]\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\left[1 -\right.}{\int[d\phi]\left[1 -\right.}$   $\left.+\ldots\right]e^{-H_0}$ (105)

$= \dfrac{\langle\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\rangle_0 - \langle\phi_{\mathbf{k}}\phi_{\mathbf{k}'}}{1 - \langle}$   $\rangle_0 + \ldots$ / $\rangle$ (106)

$= \dfrac{— - 3}{1 - 3}$  $- 12$  $+ \ldots$ /  $+ \ldots$ (107)

$\simeq — - 12$  (108)

I note that the disconnected graph, 3 from the denominator cancelled the disconnected parts of the numerator. This is a general behavior, as can be seen from the fact that the normalizing denominator $Z(h)$ arises from a derivative of $W(h) = \ln Z(h)$, whose derivatives generate only connected contributions to an n-point correlation function.

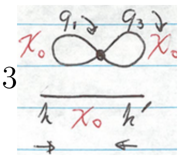
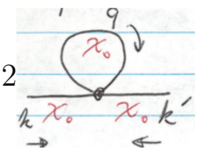
Let us now examine in more detail the actual mathematical expressions graphically representated by the diagrams, 3 + 12, appearing in the numerator, above. As seen from above perturbative expansion, these diagrams represent a numerator correction $\delta N(k)$ to the 2-point correlation function, which by translational invariance is expected to have a momentum-conserving form,

$$
\begin{aligned}
(2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')\delta N(k) &\equiv \frac{u}{Z_0}\int[d\phi]\phi_{\mathbf{k}}\phi_{\mathbf{k}'}\int_{\mathbf{q}_1\ldots\mathbf{q}_3}\phi_{\mathbf{q}_1}\phi_{\mathbf{q}_2}\phi_{\mathbf{q}_3}\phi_{-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3}e^{-\frac{1}{2}\int_{\mathbf{q}}(Kq^2+t)|\phi_{\mathbf{q}}|^2}, \\
&= 3u\tilde{G}_0(k)(2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')\int_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3}\tilde{G}_0(q_1)\tilde{G}_0(q_3)(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2)(2\pi)^d\delta^d(\mathbf{q}_3-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \\
&+ 12u\tilde{G}_0(k)\tilde{G}(k')\int_{\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3}\tilde{G}_0(q_3)(2\pi)^d\delta(\mathbf{k}+\mathbf{q}_1)(2\pi)^d\delta(\mathbf{k}'+\mathbf{q}_2)(2\pi)^d\delta(\mathbf{q}_3-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3), \\
&= (2\pi)^d\delta^d(\mathbf{k}+\mathbf{k}')\left[3u\tilde{G}_0(k)(2\pi)^d\delta^d(0)\int_{\mathbf{q}_1\mathbf{q}_3}\tilde{G}_0(q_1)\tilde{G}_0(q_3) + 12u\tilde{G}_0(k)\tilde{G}(-k)\int_{\mathbf{q}_3}\tilde{G}_0(q_3)\right].
\end{aligned}
\tag{109}
$$

Thus, we find the correction to the numerator to be given by,

$$
\delta N(k) = 3uV\tilde{G}_0(k)\left[\int_{\mathbf{q}}\tilde{G}_0(\mathbf{q})\right]^2 + 12u\tilde{G}_0(k)\tilde{G}(-k)\int_{\mathbf{q}}\tilde{G}_0(\mathbf{q}), \tag{110}
$$

$$
= 3uV\tilde{G}_0(k)G_0(\mathbf{x}=0)^2 + 12u\tilde{G}_0(k)\tilde{G}(-k)G_0(\mathbf{x}=0), \tag{111}
$$

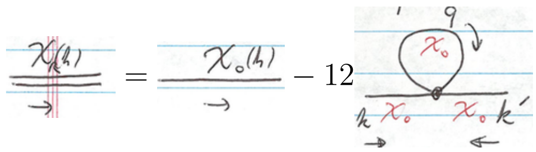
$\equiv 3$  $+ 12$  (112)

where loops represent momentum $\mathbf{q}$ integrals of propagators, $G_0(\mathbf{q})$. For later reference I note that it can be easily shown that the factor of 12 generalizes to a factor of $4(n+2)$ in the O(n) model.

Now taking into account the normalizing denominator (the partition function with the loop correction that it experiences), that cancels the disconnected diagram in the numerator, we find the correction to propagator (in graphs $\chi(k)$ denotes $G(k)$):

$$\tilde{G}(k) \approx \tilde{G}_0(k) - 12u\tilde{G}_0(k)\tilde{G}(-k)\int_{\mathbf{q}} \tilde{G}_0(\mathbf{q}), \tag{113}$$

$$= \frac{1}{Kk^2+t} - \frac{12u}{(Kk^2+t)^2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2+t}, \tag{114}$$



(115)

Looking at the inverse propagator to the same one-loop order, we find,

$$\Gamma^{(2)}(k) \equiv \tilde{G}^{-1}(k) \equiv K_R k^2 + \chi^{-1}, \tag{116}$$

$$= \tilde{G}_0^{-1}(k) + 12u\int \frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2+t}, \tag{117}$$

where the factor of 12 arises from 6 distinct ways to generate one loop contractions from a 4-point vertex, with additional factor of 2 coming from two external legs. From this correction to $G^{-1}(k)$ that is often referred to as the "self-energy", $\Sigma(k)$, we can identify the non-renormalized of the stiffness K_R

$$K_R = K, \text{ to one loop order} \tag{118}$$

(because to this order $\Sigma(k) =$  is k-independent), and the fluctuation-corrected inverse linear susceptibility $\chi(t) \equiv G(k=0)$ is given by

$$\chi^{-1}(t) = t + 12u\int \frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2+t}, \tag{119}$$

To evaluate the integral loop correction to $\chi^{-1}(t)$, I first note that its behavior depends *qualitatively* on the range of dimensionality of space, d, and thus different ranges of d must be analyzed separately.

3. *Above the upper-critical dimension, $d > 4$: mean-field theory*

I first note that for $d > 4$ the integral converges in a small t limit, and can in fact be Taylor-expanded (at least) to *linear* order in t, giving,

$$\chi^{-1}(t_R) = t + \frac{12u}{K}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2} - t\,\frac{12u}{K^2}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^4} + \frac{12u}{K}O[(t/K)^{d/2-1}], \tag{120}$$

$$= t + \frac{12uC_d\Lambda^{d-2}}{K(d-2)} - t\,\frac{12uC_d\Lambda^{d-4}}{K^2(d-4)} + \frac{12u}{K}O[(t/K)^{d/2-1}], \tag{121}$$

$$\equiv t_R\left(1 - \frac{12uC_d\Lambda^{d-4}}{K^2}\right) + \frac{12u}{K}O[(t_R/K)^{d/2-1}]. \tag{122}$$

In above, $\Lambda = 2\pi/a$ is the UV momentum cutoff (implicit in all integrals when necessary), set by the underlying lattice constant and used to cutoff the (for $d > 4$ UV divergent) integrals and $C_d = S_d/(2\pi)^d = \frac{1}{2^{d-1}(\pi)^{d/2}\Gamma(d/2)}$, with S_d a surface area of a d-dimensional ball ($S_1 = 2, S_2 = 2\pi, S_3 = 4\pi$, etc.).[11] I note that third correction term is valid for $4 < d < 6$, giving a nonanalytic correction $\sim t^{d/2-1} \ll t$, that at small t is subdominant to the first two leading terms. For $d > 6$, it can be checked that the correction instead scales as t^2, with the subdominant nonanalyticity arising at even higher order in t. In the last equality above, I defined the renormalized reduced temperature

$$t_R \equiv t + \frac{12uC_d\Lambda^{d-2}}{K(d-2)}, \tag{123}$$

shifted upwards by the first t-independent constant term. In the final equality I also replaced t by the renormalized (upward-shifted) reduced temperature t_R, valid to lowest order in the small quartic coupling u. Recalling that $t = a_0(T - T_c)$, we find that these fluctuation corrections *suppress downward* the critical transition temperature from $T_c \to T_c^R$,

$$T_c^R = T_c - \frac{12uC_d\Lambda^{d-2}}{Ka_0(d-2)}. \tag{124}$$

This is consistent with the physical expectations, that fluctuations tend to suppress the ordered phase, enhancing the range of the disordered phase.

4. *Below the upper-critical dimension, $2 < d < 4$: breakdown of mean-field theory*

Examining the integral correction to χ^{-1} in (119) of the previous subsection, it is easy to see that for $2 < d < 4$ the first constant correction to t remains infrared (IR) convergent and therefore still non-divergently upward-shifts $t \to t_R$ as for $d > 4$ of the previous subsection. However, in qualitative contrast, the second correction, linear in t diverges in IR for $d < 4$, signaling the breakdown of the above perturbative expansion in small t. This is true no matter how small u is and how low the absolute temperature T is.

To assess the behavior of $\chi^{-1}(t_R)$, I first shift t to t_R by adding and subtracting the constant shift integral $\delta t = 12u\int \frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2} = \frac{12uC_d\Lambda^{d-2}}{K(d-2)}$, (IR convergent for $d > 2$) thereby obtaining an integral that diverges in the IR for a vanishing t_R,

$$\chi^{-1} \approx t_R + 12u\int\frac{d^dq}{(2\pi)^d}\left[\frac{1}{Kq^2+t}-\frac{1}{Kq^2}\right], \tag{125}$$

$$\approx t_R - \frac{12u}{K^2}\int\frac{d^dq}{(2\pi)^d}\frac{t_R}{q^2(q^2+t_R/K)}, \tag{126}$$

$$\approx t_R\left[1-\frac{12uC_d}{K^2}\int_0^\infty dq q^{d-3}\frac{1}{q^2+t_R/K}\right], \tag{127}$$

$$\approx t_R\left[1-(K/t_R)^{(4-d)/2}\frac{6\pi uC_d}{K^2|\sin(\pi d/2)|}\right], \tag{128}$$

$$\approx t_R\left[1-\text{const.}ut_R^{-(4-d)/2}+\ldots\right], \quad \text{for } d<4. \tag{129}$$

I note, that, although superficially the correction term in (126) is proportional to t_R, the corresponding integral is IR divergent, and therefore is required to be performed for a small but nonzero t_R, giving nonanalytic expansion in t_R. Complementing a direct calculation in the last equality, above, a power-counting easily shows that the integral scales as $\sim 1/q^{4-d}$ and thus must grow (diverge) in IR as $\xi_0^{4-d} \sim (t_R/K)^{-(4-d)/2}$. Since for small t_R this and the second term in (129) diverge, sufficiently close to T_c^R the perturbation theory in u breaks down no matter how small the quartic coupling is. This breakdown happens sufficiently close to T_c^R, for $t_R < t_R^{Ginzburg}$, where

$$t_R^{\text{Ginzburg}} = \left[\frac{6\pi uC_d}{K^{d/2}|\sin(\pi d/2)|}\right]^{2/(4-d)} \sim (u^2/K^d)^{1/(4-d)}. \tag{130}$$

We note that this is precisely the Ginzburg criterion, found in previous set of lecture notes 3 on breakdown of Landau theory. As we will find in the next lecture, this *non-analytic* expansion (breakdown of perturbation theory) in t_R is a generic signature of the expected modification of the mean-field exponent $\gamma_{MF} = 1$ inside the Ginzburg region, where exponents can be calculated using the renormalization group (RG).

5. *Below the lower-critical dimension, $d < 2$: instability of the ordered phase*

Finally, examining the integral correction δt for $d < 2$ in Eq.(122), I find that indeed it diverges in the IR. From (124), we thus find that the downward suppression of T_c^R diverges. This suggests (but must be reexamined more carefully by a more reliable ordered state analysis) that the lower-critical dimension of $d_{lc} = 2$, below which the phase transition is destroyed by thermal fluctuations.

6. *Graphical correction to the quartic vertex*

As another example, let us also look at another 4-point correlation functions. We focus on a “truncated” 1PI correlator in momentum space,

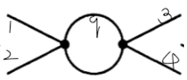
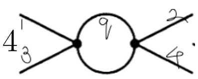
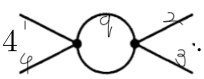
$$\Gamma^{(4)} \equiv \mathrm{C}_{tr}^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle_{tr} \tag{131}$$

where “truncation” corresponds to not including external propagator legs, by dividing out the external propagators,

$$C_{tr}^{(4)} = C_{conn}^{(4)} \times G_0^{-1}(k_1) G_0^{-1}(k_2) ... G_0^{-1}(k_4). \tag{132}$$

Above, $C_{conn}^{(4)}$ is the ”connected” correlation function, i.e., the one that excludes contributions to the correlator that can be written as product of lower order correlation functions. As will become more clear in the next set of lectures on renormalization group, the significance of such a correlator is that it determines the effective “renormalized” quartic coupling u_R, characterizing the theory in the presence of fluctuations.

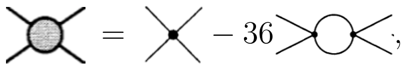
$$\begin{aligned} C_{conn}^{(4)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= G_0(\mathbf{k}_1) \dots G_0(\mathbf{k}_4) (2\pi)^d \delta^d(\mathbf{k}_1 + \dots + \mathbf{k}_4) \\ \times & \left[4! u - \frac{u^2}{2!} 6^2 \cdot 2 \cdot 2^2 \cdot 2 \left(\int_{\mathbf{q}} \frac{1}{(Kq^2 + t)(K(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{q})^2 + t)} \right. \right. \\ + & \left. \left. \int_{\mathbf{q}} \frac{1}{(Kq^2 + t)(K(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{q})^2 + t)} + \int_{\mathbf{q}} \frac{1}{(Kq^2 + t)(K(\mathbf{k}_1 + \mathbf{k}_4 - \mathbf{q})^2 + t)} \right) \right], \end{aligned} \tag{133}$$

$= 4!$  $- 72 \cdot 4$  $+ 72 \cdot 4$  $+ 72 \cdot 4$  . (134)

Observe the combinatorial factors: (i) the 4! comes from the total number of ways to connect external fields $\phi_{\mathbf{k}_1}, \dots, \phi_{\mathbf{k}_4}$ to the quartic vertex u, (ii) factors of 6 come from the number of ways to select 2 legs (from each quartic vertex) that will connect as internal legs of the loop, and a factor of 2 for the two ways of connecting them up into a loop, (iii) the other factor of $2^2 \cdot 2$ are the number of ways to connect the fields $\phi_{\mathbf{k}_1}, \phi_{\mathbf{k}_2}$ on one side and $\phi_{\mathbf{k}_3}, \phi_{\mathbf{k}_4}$ on other side of the resulting one-loop diagram, (iv) there are total of three channels (referred to as s, t and u in particle physics) corresponding to three ways of pairing the four external momenta together fields to the diagram.

Focussing on the zero momentum limit of this 4-point vertex (physically representing 2-particle scattering in quantum field theory), reduces three scattering channels into an identical form, resulting in overall 4! factor as for the bare vertex (first) term, with which we find,

$$\frac{1}{4!}\Gamma^{(4)}(\{k_i = 0\}) \equiv u_R \;=\; u\left(1 - 36u\int^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{(Kq^2+t)^2}\right), \tag{135}$$



(136)

$$= u\left(1 - \frac{36uC_d}{K^2}\int_0^{\Lambda} dq\frac{q^{d-1}}{(q^2+\xi_0^{-2})^2}\right), \tag{137}$$

$$= u\left(1 - \xi_0^{(4-d)}\frac{36uC_d}{K^2}\int_0^{\Lambda\xi_0} dq'\frac{q'^{d-1}}{(q'^2+1)^2}\right), \tag{138}$$

$$= u\left(1 - \frac{36uC_d}{K^2}\times\begin{cases}\Lambda^{d-4}, & \text{for } d>4,\\ a_d\xi_0^{4-d}\sim t^{-(4-d)/2}, & \text{for } d<4.\end{cases}\right), \tag{139}$$

$$= u\left[1-\left(\frac{\xi_0}{\xi_G}\right)^{4-d}\right], \text{ for } d<4, \tag{140}$$

where $a_d = \frac{\pi(d-2)}{4|\sin(\pi d/2)|}$ is a constant, and to compute the integral I rescaled the momenta of integration by ξ_0. In the last line I wrote the correction in the explicitly dimensionless form, defining the Ginzburg length,

$$\xi_G = \left(\frac{K^2}{36ua_cC_d}\right)^{1/(4-d)}, \tag{141}$$

relative to which the correlation length ξ_0 is measured, and beyond which the nonlinearity u begins to matter. For later reference I note that it can be easily shown that the coefficient 36 above generalizes to a factor of $4(n+8)$ in the O(n) model.

The crucial observation above is that in $t\to 0$ (or equivalently, $\xi_0\to\infty$) limit, the one-loop integral has qualitatively distinct behaviors for $d>4$, where it is IR convergent, and for $d<4$, where it is IR divergent and thus sensitively and nonanalytically depends on t and ξ_0,

$$\frac{\delta u_R}{u}\sim\frac{u}{K^2}\begin{cases}\Lambda^{d-4}, & \text{for } d>4,\\ \xi_0^{4-d}, & \text{for } d<4.\end{cases} \tag{142}$$

Although calculated exactly above, these asymptotic behaviors are also straightforwardly obtained by power-counting (dimensional analysis).

Once again, as with the 2-point correlation function we observe that for $d > d_{uc} = 4$ the one-loop fluctuation-correction to u_R is finite near the critical point T_c, as $t\to 0$. In qualitative contrast, for $d < d_{uc} = 4$ this one-loop correction diverges near the critical point ($t\to 0$), leading to a breakdown of the perturbative expansion, no matter how small u and

T are. The criterion for this breakdown, i.e., the correction $\frac{\delta u_R(t_G)}{u} = 1$, then again leads to the Ginzburg criterion in (130).

These findings then lead to a key question: What do we do to describe an interatcting fluctuating system near its critical point, T_{cR} ($t_R \to 0$)? We will find the answer in the next set of lectures on RG, as discovered by Ken G. Wilson, and many among contributers.[4, 7–10]

---

[1] Pathria: *Statistical Mechanics*, Butterworth-Heinemann (1996).

[2] L. D. Landau and E. M. Lifshitz: *Statistical Physics*, Third Edition, Part 1: Volume 5 (Course of Theoretical Physics, Volume 5).

[3] Mehran Kardar: *Statistical Physics of Particles*, Cambridge University Press (2007).

[4] Mehran Kardar: *Statistical Physics of Fields*, Cambridge University Press (2007).

[5] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman : *The Theory of Critical Phenomena*, Oxford (1995).

[6] John Cardy: *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics.

[7] P. M. Chaikin and T. C. Lubensky: *Principles of Condensed Matter Physics*, Cambridge (1995).

[8] M. E. Fisher, *Rev. Mod. Phys.* **42**, 597 (1974).

[9] K. G. Wilson and J. Kogut, *Phys. Rep.* **12** C, 77 (1974).

[10] J. Zinn-Justin: *Quantum Field Theory and Critical Phenomena*, Oxford (1989).

[11] see appendix A in Pierre Le Doussal, Leo Radzihovsky, "Anomalous elasticity, fluctuations and disorder in elastic membranes", arxiv.org/pdf/1708.05723.pdf.

[12] *Quantum Field Theory of Many-body Systems*, Xiao-Gang Wen.

[13] *Path Integrals*, R. P. Feynman and Hibbs. plore properties of this

[14] *Principles of Condensed Matter Physics*, by P. M. Chaikin and T. C. Lubensky.

[15] L.D. Landau, Phys. Z. Sowjetunion **II**, 26 (1937); see also S. Alexander and J. McTague, Phys. Rev. Lett. **41**, 702 (1984).

[16] R.E Peierls, Ann. Inst. Henri Poincaré **5**, 177 (1935); L.D. Landau, Phys. Z. Sowjetunion II, 26 (1937)

[17] N.D. Mermin and H. Wagner, "Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models", *Phys. Rev. Lett.* **17**, 1133-1136 (1966); P.C. Hohenberg, "Existence of Long-Range Order in One and Two Dimensions", *Phys. Rev.* **158**, 383, (1967); N.D. Mermin, *Phys. Rev.* **176**, 250 (1968).

[18] S. Sachdev, "Quantum phase transitions" (Cambridge University Press, London, 1999).